# Berkeley Math Circle Monthly Contest # 6

Solutions

1. *Find all integer solutions to*

$$x^2 - 3y^2 = 17.$$

Consider the equation modulo 3. Modulo 3, $0^2 \equiv 0$, $1^2 \equiv 1$, and $2^2 \equiv 1$, so the squares modulo 3 are only 0 and 1. However, the equation modulo 3 gives $x^2 \equiv 2$. Thus there are no integer solutions to the equation.

2. *The point $P$ lies inside an equilateral triangle and its distance to the three vertices are $3, 4, 5$. Find the area of the triangle.*

**Answer:** $\frac{25}{4}\sqrt{3} + 9$

We will call the equilateral traingle $ABC$ with $|PA| = 3$, $|PB| = 4$ and $|PC| = 5$. Rotate $\triangle ABC$ around point $A$ $\pi/3$ radians so that the image of point $B$ is point $C$. Let the image of point $C$ be $C'$ and the image of point $P$ be $P'$. Then, since it is a rotation, $|AP| = |AP'| = 3$ and $\angle PAP' = \pi/3$, so $\triangle PAP'$ is equilateral. Thus $|PP'| = 3$. Since $C$ is the image of $B$ under the rotation, $|P'C| = 4$. Thus $\triangle PP'C$ is a right triangle with sides 3,4,5. So $\angle PP'C = \pi/2$, and $\angle AP'C = \angle AP'P + \angle PP'C = 5\pi/6$. By law of cosines on $\triangle AP'C$, $AC^2 = AP'^2 + P'C^2 - 2AP' \cdot P'C \cos 5\pi/6 = 25 + 12\sqrt{3}$. Thus the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}(25 + 12\sqrt{3}) = \frac{25}{4}\sqrt{3} + 9$.

3. *Suppose that $n = a^2 + b^2 + c^2$ where $a, b, c$ are positive integers. Prove that for any positive integer $k$, $n^{2k}$ can be written $A^2 + B^2 + C^2$, where $A, B, C$ are positive integers.*

We will prove the claim by induction. First, we know that

$$n^2 = (a^2 + b^2 + c^2)^2 = (a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2.$$

Then, suppose that $n^{2k} = A^2 + B^2 + C^2$ for some positive integers $A, B, C$. Then

$$n^{2k+2} = n^2 \cdot n^{2k} = (nA)^2 + (nB)^2 + (nC)^2.$$

Since $n, A, B, C$ are all positive integers, $nA, nB, nC$ are all also positive integers, so $n^{2k+2}$ can also be written as the sum of squares of three positive integers.

4. *You are placing letters from an alphabet of* $n$ *letters around a circle by the following rules:*

   *(i) No two adjascent letters are the same*

   *(ii) For any two different letters* $a$ *and* $b$ *it is possible to draw a line through the circle such that all of the* $a$*'s are on one side, and all of the* $b$*'s are on the other side.*

   **Answer:** $2n - 2$

   We will prove this by induction. First node that condition (ii) is equivalent to stating that there cannot be a pattern that goes $a - b - a - b$ (with different instances of the letters) around the circle. If there was such a pattern, a line through the circle would have to intersect the circle between the letters, which means that it would have to intersect the circle four times, which is clearly impossible.

   From this we will proceed with induction. Suppose $n = 2$. Then we can only put 2 letters on the circle, which confirms the hypothesis. Assume that for all $1 < n \leq k$ we can only place $2k - 2$ letters around the circle, and consider $n = k + 1$. Because we can get at least $2k - 2$ letters onto the circle (by not using the $k + 1$st) there will be at least one letter (without loss of generality, $a$) which appears at least twice. Connect two instances of the $a$ with a line. We know that there is some letter $b$ which appears on one side of this line but not the other. Now consider two circles, one with the letters to the left of the $a$'s (in the same order) and then one $a$ ($k_1$ distinct letters), and one with the letters to the right of the $a$ (in the same order) and then one $a$ ($k_2$ distinct letters). We know that $k_1, k_2 > 1$, and that $k_1 + k_2 = k + 2$ (since the $a$ is counted on both). Thus on the first circle there can be at most $2k_1 - 2$ letters, and on the second circle there can be at most $2k_2 - 2$ letters. All together, there are $2k_1 - 2 + 2k_2 - 2 = 2(k_1 + k_2) - 4 = 2(k+2) - 4 = 2(k+1) - 2$ letters, as desired.

5. *Prove that the boundary of the intersection of $n$ circles consists of at most $2n - 2$ circular arcs.*

   This problem is equivalent to the previous one. To see this, take each of the $n$ circles and assign it a letter. Then for each boundary arc of the intersection, assign it the letter belonging to the circle that it is a part of. Condition (i) is satisfied because if two adjacent arcs belong to the same circle they are unified into one arc. Condition (ii) is satisfied because two distinct circles can only intersect at two points. Thus the intersection can have at most $2n - 2$ circular arcs as the boundary.