Berkeley Math Circle
Monthly Contest #5 Solutions

1. Consider a polyhedron with n vertices. Since a polyhedron is a three dimensional figure, the smallest number of edges emanating from a vertex is 3. Also, since there are n vertices the greatest number of edges emanating from a vertex is $n-1$. By the pigeonhole principle, since there are $n-3$ possible degrees for a vertex and n vertices, there will be at least two with the same degree.

2. Consider any line in the plane not going through any of the given points. By the extended pigeonhole principle, there will be at least 7 points on one side of the line. Consider the following subsets of the points: $\{P_1, P_2\}$, $\{P_3, P_4\}$, $\{P_5, P_6\}$, $\{P_7, P_8\}$, $\{P_9, P_{10}\}$, $\{P_{11}, P_{12}\}$, $\{P_{13}\}$. By the pigeonhole principle (since there are 7 sets that partition the points) on that side of the line there will either be one point in each of the sets or two points in some set.
   Case 1: There are two points in some set. Then the segment connecting those two points does not intersect the line.
   Case 2: There is one point in each set. Then P_13 is on the side of the line with more points. If P_{12} or P_1 is on that same side then we have two consecutive points on the same side of the line, which is Case 1. Suppose they are not. Thus P_{11} and P_2 are on the same side of the line as P_{13}. By the same logic, P_{10} and P_3, P_9 and P_4, P_8 and P_5 and P_6 and P_7 are on the same side of the line. However, then we have two consecutive points on the same side of the line, so the segment connecting those two points does not intersect the line.

3. Suppose $x = y$. Then $f(\sqrt{2}x) = f(x)^2$. Also, $f(2x) = f(\sqrt{2}\sqrt{2}x) = f(x)^4$. Let $f(1) = k$. Then $f(2) = k^4$, $f(4) = k^{16}$, and in general (by induction) $f(x2^n) = f(x)^{4^n}$. Analogously, $f(x) = f\left(\frac{x}{\sqrt{2}}\right)^2$ and $f(x) = f\left(\frac{x}{2}\right)^4$, so in general $f(x2^{-n}) = f(x)^{\frac{1}{n^4}}$. To get $f(2^{\frac{p}{q}})$ simply calculate $f(2^p)$ and let $x = 2^p$. Then calculate $f(2^{-q}x)$. Thus we have the general formula $f\left(2^{\frac{p}{q}}\right) = k^{4^{\frac{p}{q}}}$.

   Since $f(x)f(-y) = f(\sqrt{x^2+y^2}) = f(x)f(y)$ we know that $f(y) =$

$f(-y)$. Thus, since the rationals are dense in the reals so is $2^{\frac{p}{q}}$ in the positive reals, we can get the formula for the function: $f(x) = k^{x^2}$.

Plug the function back into the functional equation. $f(\sqrt{x^2+y^2}) = k^{x^2+y^2} = k^{x^2}k^{y^2} = f(x)f(y)$. Thus the function works, and so it is the solution to the functional equation.

4. We shall proceed using complex numbers. In this venue of attack, we coordinize the plane with 0, the circumcenter of $\triangle ABC$, as the origin, —AO— = 1, and represent each of the points A, B, C, D as a complex numbers a, b, c, d. The real and complex parts of each number represent the coordinates of its respective point. as a, b, c are on a circle about the origin and $A0 = 1$, we have $|a| = |b| = |c| = 1$. Letting g be a complex number representing the centroid of $\triangle ABC$, we have $(a+b+c)/3 = g$. But, in any triangle, the circumcenter (O), orthocenter (H), and centroid (G) are collinear and $OH = 3OG$, G between O and H. Thus, as O is the origin, $h = 3g = 3(a+b+c)/3 = a+b+c$. We are given that $\angle DAB = \angle ABC = \angle BCD$, and we wish to encode this in our complex numbers. First note that is really a statment about the vectors $d-a$, $b-a$, etc... Expressing the vectors in polar coordinates, we notice the angle from $d-a$ to $b-a$ is really the difference in their polar angles. Dividing the two $\frac{d-a}{b-a}$ yeilds a vector of the proper direction, but still has a nonunit length, and thus cannot be used for comparison. However, if we double are angle, thus essentially working modulo π instead of 2π we only add possible solutions to the equations, we lose none. Thus, dividing by the conjugate, and thus caputring only the angle we find.

$$\frac{d-a}{b-a}\frac{\bar{b}-\bar{a}}{\bar{d}-\bar{a}} = \frac{a-b}{c-b}\frac{\bar{c}-\bar{b}}{\bar{a}-\bar{b}} = \frac{b-c}{d-c}\frac{\bar{d}-\bar{c}}{\bar{b}-\bar{c}}.$$

But, $a\bar{a} = 1$ (and similarly for b, c), so $\bar{a}-\bar{c} = \frac{1}{a} - \frac{1}{c} = \frac{c-a}{ac}$. So, $(d-a)c = ab(1-a\bar{d})$ and $(d-c)a = cb(1-c\bar{d})$. Multplying through by c^2 and a^2 respectivly, and subtracting yeilds $(d-a)c^3-(d-c)a^3 = c^2ab(1-a\bar{d})-a^2cb(1-c\bar{d})$. Thus, $d(c^3-a^3) = ac(c^2-a^2)+ac(bc-abc\bar{d}-ab+abc\bar{d})$ and then $d(c-a)(c^2+a^2+ac) = (c-a)ac(c+a+b)$. Finally, we have $d = (a+b+c)(\frac{ac}{a^2+ac+c^2})$.

Now we have calculated are points O as 0, H as $h = a+b+c$ and D as $d = (a+b+c)(\frac{ac}{a^2+ac+c^2})$ and we return to the original problem of proving O,H, and D are collinear. as O is the origion, this is equivlent to proving that d is a real multiple of h; i.e. that $d/h = \frac{ac}{a^2+ac+c^2}$ is real. But $(\frac{ac}{a^2+ac+c^2})(\frac{\bar{a}^2+\bar{c}^2}{\bar{a}^2+\bar{c}^2}) = \frac{ac\bar{a}^2\bar{c}^2}{a^2\bar{a}^2\bar{c}^2+ac\bar{a}^2\bar{c}^2+c^2\bar{a}^2\bar{c}^2} = \frac{\bar{a}\bar{c}}{\bar{a}^2+\bar{a}\bar{c}+\bar{c}^2}$. Thus, $\frac{ac}{a^2+ac+c^2}$ equals its conjugate and is therefore real as desired.

5. The only position for the single -1 is the center of the 5×5 grid.

Note: The only squares we can use to toggle the grid are 2×2, 3×3, 4×4, and 5×5. We shall say "use $a(b,c)$" to denote toggling all the $-1/+1$ values in the $a \times a$ square with lower left hand square at coordinate (b,c) where the lower left hand coordinate of the 5×5 square is $(1,1)$ and the uppper right (which can never be used of course) is $(5,5)$.

I. A -1 in the center position can be removed. By using the squares $3(1,1)$,$3(3,3)$,$2(1,4)$,$2(4,1)$,$5(1,1)$.

II. A single -1 anywhere else cannot be removed. Let us consider the value S, the sum of the values in our grid modulo 4. Clearly, the grid begins with $S = -1 + 24 \cdot 1 = 23 = 3$ and the grid must end with $S = 25 \cdot 1 = 1$. Let us now investigate how each of the square transformations modifies the value of S for our grid. First notice that each time a value is "toggled" from -1 to $+1$ or $+1$ to -1, $S' = S - (-1) + 1 = S + 2$ or $S' = S - 1 + (-1) = S - 2 = S + 2$, and so the value of S is always increased by exactly 2 (modulo 4). If we use $a(b,c)$, we toggle an entire $a \times a$ group of squares, or $a \cdot a$ squares. Then, if a is even, so is $a \cdot a$, and we have $S' = S + 2 \cdot a \cdot a = S + 2 \cdot 2k = S$; that is, S is fixed, by such a square toggle. However, if a is odd, so is $a \cdot a$, and we have $S' = S + 2 \cdot a \cdot a = S + 2 \cdot (2k+1) = S + 2$. Thus, S changes only if we use and odd square, and then by exactly 2. Now, suppose we find a some series of toggles that takes us from our intial condtion of a single -1 to our final condition with no -1s. Let x be the number of even squares used, and y be the number of odd squares. Then, $1 = S' = S + 0x + 2y = S + 2y = 3 + 2y$. Clearly, y must be odd. However, notice that the only odd sqares we can use are 3×3 and 5×5, and both of these always toggle the center square, regardless of position. But, as we use the odd squares in odd number of times, the center square must be toggled an odd number of times. So, if the center square began as $+1$ it is now a -1, and thus we must not be our final configuration after using our sequence of squares, as we supposed.

Therefore, if the -1 is placed anywhere, but the center it cannot be extricated by any sequence of these squares.