Berkeley Math Circle
Monthly Contest
Due January 6

1. Prove that on any polyhedron there are two vertices with the same degree. The degree of a vertex is the number of edges eminating from that vertex.

2. Let thirteen points $P_1, P_2, \ldots, P_{13}$ be given in the plane, and suppose they are connected by the segments $P_1P_2, P_2P3, \ldots, P_{12}P_{13}, P_{13}P_1$. Is it possible to draw a straight line which passes through the interior of each of these segments?

3. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ that, for all x and y, satisfy the functional equation
$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

4. Let $ABCD$ be a convex quadralateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O be the orthocenter and circumcenter of $\triangle ABC$. Prove H, O and D are collinear.

5. A 5×5 square has been dissected into a unit square grid. One of the unit squares has been filled with "-1" and the rest have been filled with "$+1$". In each step, one can choose a square, with side at least 2 and its sides parallel to the sides of the grid, and change all the signs in the square. Find all terh possible intial format(s) such that it is possible to get an all "$+1$" grid in a finite number of steps.