

Berkeley Math Circle  
Monthly Contest  
Due January 6

1. Find the smallest prime divisor of

$$4^{1001001001} + 1001001001^4.$$

2. Given that

$$x + x^{-1} = \frac{1 + \sqrt{5}}{2},$$

find  $x^{2000} + x^{-2000}$ .

3. Berkeley's professors are holding a New Year's party this winter and  $n$  people decide to attend. During the main feast of the celebration the  $n$  guests need to sit around a round table – each person sits next to exactly two other people. However, some of the guests hate other people at the party. If person  $A$  hates person  $B$  then person  $A$  and person  $B$  cannot sit next to each other at the table for the grand feast. We say the guests have a spite level of  $m$  if the list of all pairs  $(A, B)$ , where person  $A$  hates person  $B$ , has length  $m$ . What is the maximum spite level the group can have and still be guaranteed to find an agreeable seating arrangement for the New Year's feast?
4. Given real  $a, b, c, d, e > 0$  such that  $a + b + c + d + e = 1$ , prove

$$\frac{a^6}{(b^2 + d^2 + e^2)^2} + \frac{b^6}{(c^2 + e^2 + a^2)^2} + \frac{c^6}{(d^2 + a^2 + b^2)^2} + \frac{d^6}{(e^2 + b^2 + c^2)^2} + \frac{e^6}{(a^2 + c^2 + d^2)^2} \geq \frac{1}{45}.$$

5. Let  $r$  be the inradius and  $R$  the circumradius of  $\triangle ABC$ . Also, let  $[ABC]$  denote the area of  $\triangle ABC$ . Let  $P, Q, R$  be the points of tangency of the incircle of  $\triangle ABC$  to the sides  $BC, AC$ , and  $AB$ , respectively, and  $X$  be the point of intersection of  $AP$  and  $BQ$ . If  $\frac{R}{r} = 3$  and  $[ABC] = 6$  prove that

$$2[AXQ] \cdot [BXR] \cdot [CXP] + [CQX] \cdot [ARX] \cdot [BPX] \leq 2.$$