Functional Equations & Recurrence Relations

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(This is the first of two presentations on this topic)

1 Quick Outline

A *recurrence relation* gives the values of a sequence in terms of its previous values; a *functional equation* gives values of a function in terms of other values of the function. (Note: a sequence *is* a function, too! Recurrence relations are a special kind of functional equation.)

Usually, the goal is to find a *closed form* expression for the sequence or function; sometimes you want to find a specific value; occasionally, there's something else to do.

Working with recurrence relations often involves induction in some form, though it is frequently possible to find closed form solutions without directly using induction.

Many situations can be recast in terms of a recurrence relation or functional equation. This is especially true of combinatorial problems.

2 Easy illustrative examples

2.1 (AHSME 1999, #13) Define a sequence of real numbers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and $a_{n+1}^3 = 99a_n^3$ for all $n \geq 1$. Then a_{100} equals? *The original problem was multiple choice.*

2.2 (AHSME 1999, #20) The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 19$, $a_9 = 99$, and for all $n \geq 3$, a_n is the arithmetic mean of the first $n-1$ terms. Find a_2. *The original problem was multiple choice.*

2.3 (AHSME 1998, #17) Let $f(x)$ be a function with the two properties:

(a) for any two real numbers x and y, $f(x+y) = x + f(y)$, and

(b) $f(0) = 2$.

What is the value of $f(1998)$? *The original problem was multiple choice.*

2.4 (AHSME 1997, #27) Consider those functions f that satisfy $f(x+4) + f(x-4) = f(x)$ for all real x. Any such function is periodic and there is a least common positive period p for all of them. Find p. *The original problem was multiple choice.*

2.5 (Common idea) The probability a team wins its next game is .75 if it won its last game and .25 if it lost its last game. What's the probability a team that wins game 1 will win game 10?

2.6 (Common) Into how many pieces can a pizza be divided by n straight vertical cuts? (Assume the pizza is essentially 2-dimensional – also convex. And no moving the pieces around between the cuts.)

2.7 (Variations of the pizza problem)

1. Into how many pieces can a cake be cut with n straight cuts (not necessarily vertical! The point is that the cake has thickness, so now the shape is 3-dimensional and the cuts are not lines, but planes!)

2. Go back to the essentially two-dimensional pizza – but now assume the cuts are not straight lines, but V-shaped (that is, a cut is made with a "wedger" – starting from a point, it generates two rays). How many

3. Go back to the two-dimensional pizza and n straight line cuts, but now count the maximum number of pieces that don't have any of the crust on the boundary.

3 Basic examples, famous examples

- $a_{n+1} = a_n + k$ (So $a_n = a_1 + (k-1)n$)
- $a_{n+1} = a_n + n$ (So $a_n = a_1 + 1 + 2 + \ldots + (n-1) = a_1 + n(n-1)/2$)
- $a_{n+1} = a_n \cdot k$ ($a_n = a_1 k^{n-1}$)
- $a_{n+1} = a_n \cdot n$ ($a_n = a_1(n-1)!$)
- Fibonacci sequence: $a_{n+1} = a_n + a_{n-1}$ (for $n \geq 2$), $a_1 = a_2 = 1$ (Closed form? various ways to express it. Discussed at the board.)
- Fibonacci variants (Discussed on the board).
- Cauchy equation: $f(x+y) = f(x) + f(y)$. ($f(x) = f(1) \cdot x$, for *rational* x. If f is defined on the reals, and is continuous, then $f(x) = f(1) \cdot x$ everywhere). There are variants of this equation involving multiplication instead of addition...
- Josephus Problem: n rebels (let's say $n = 41$ for simplicity) are trapped by the Romans and decide to kill themselves rather than be captured. They form a circle and go around it, killing every other person until one is left, who must commit suicide. As the lone spy in the group, you'd like to position yourself to be the one person left. What position do you stand in? (Note: in the original story, the 41 rebels killed every *third* person and Josephus found the right places for himself and an accomplice to stand in order to be the last two people left).

4 Basic methods of solution

(This will be expanded in later sections)

- Guess the answer, prove it by induction
- try special values, like 0 or 1
- try to fit to most common patterns (listed above)

- try geometric series solutions or polynomials (when appropriate)
- Finding simple solutions that generate all possible solutions. ("repertoire" method). This is particularly appropriate when the sum of two distinct solutions is another solution, or when solutions multiplied by a constant form another solution.

5 Fairly common types of problems

5.8 (AIME 1996) A bored student walks down a hall that contains a row of closed lockers, numbered 1 to 1024. He opens the locker numbered 1, and then alternates between skipping and opening each closed locker thereafter. When he reaches the end of the hall, the student turns around and starts back. He opens the first closed locker he encounters, then alternates between skipping and opening each closed locker thereafter. The student continues wandering back and forth in this manner until every locker is open. What is the number of the last locker he opens?

5.9 (AIME 1994) The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2 .$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

5.10 (AIME 1993) Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

5.11 (AIME 1992) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1 and that $a_{19} = a_{92} = 0$. Find a_1.

5.12 (British Math Olympiad, 1977, #1) A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied:

(a) $f(mn) = f(m) + f(n)$, for all positive integers m, and n ;

(b) $f(n) = 0$ whenever the units digit of n (in base 10) is a '3'; and

(c) $f(10) = 0$.

Prove that $f(n) = 0$, for all positive integers n.

5.13 (Putnam, 1999, problem A-1) Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x:

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0 \end{cases}$$

See Polya Contest 1995 Power Round on attached sheet.

6 Summation and recurrence

A summation $S_n = \sum_{k=1}^{n} a_k$ can be thought of as a recurrence relation, since $S_1 = a_1$ and $S_{n+1} = S_n + a_{n+1}$. Consequently, the same ideas used to find closed forms for recurrences may help find closed forms for sums (and vice versa).

Note some additional techniques useful for sums: *perturbation method (splitting a term off the sum and rewriting it)*

Some examples:

1. $\sum_{k=1}^{n}(-1)^k k^2$
2. $\sum_{k=1}^{n} k \cdot 2^k$

7 Some deeper ideas

Difference operator Δ (already mentioned in a problem above) is very useful when dealing with sequences, especially those that come from polynomials.

Think about $\Delta(x^n)$; but is is especially useful to look at *falling powers*, that is:

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

(Rising powers are similarly defined, $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$, but we won't use them here.) Also consider the *polynomial* $\binom{x}{m} = \frac{x^{\underline{m}}}{m!}$.

What is $\Delta(x^{\underline{m}})$? What is $\Delta(\binom{x}{m})$? What is $\Delta^k(x^{\underline{m}})$? $\Delta^k(\binom{x}{m})$?

The polynomial $\binom{x}{m}$ is 0 for $x = 0, 1, \ldots, m-1$ and 1 for $x = m$, so it is easy to see how its succession of finite differences will look. This gives a way to resurrect any polynomial from the difference sequence. (This is an example of the repertoire method).

This approach also gives a nice proof of the recurrence relation:

$$p(x+n) = \binom{n}{1}p(x+n-1) - \binom{n}{2}p(x+n-2) + \ldots + (-1)^{n-1}p(x)$$

for any polynomial of degree *less* than n.

(This result is given as Proposition 22, in section 4 of Gabriel Carroll's paper on polynomials presented at an earlier session of the math circle.)

8 Various Problems

Some of these are quite difficult! I can't guarantee that they are in order of difficulty; in fact, I'm rather sure they aren't. Try a few; next week, I'll go over some of them, have hints for the others, and cover some more advanced ideas.

8.14 (Manhattan (Kansas) Math Olympiad 1999) In the sequence $1, 1, 2, 3, 7, 22, 155, 3411, \ldots$ every term is equal to the product of the previous two terms plus 1. Prove that there are no terms in the sequence which are divisible by 4.

8.15 (Leningrad Math Olympiad 1988 Grade 10 Main Round) The functions $f(x)$ and $g(x)$ are defined on the real axis so that they satisfy the following condition: for any real numbers x and y, $f(x + g(y)) = 2x + y + 5$. Find an explicit expression for the function $g(x + f(y))$.

8.16 (Leningrad Math Olympiad 1987 Grade 10 Elimination Round) The continuous functions $f, g\colon [0,1] \to [0,1]$ satisfy the following condition $f(g(x)) = g(f(x))$ for every $x \in [0,1]$. It is known that f is an increasing function. Prove that there exists an $a \in [0,1]$ such that $f(a) = g(a) = a$.

8.17 (Leningrad Math Olympiad 1987 Grade 9 Elimination Round) Let (A_n) be a sequence of natural numbers such that $A_1 < 1999$ and $A_i + A_{i+1} = A_{i+2}$ for any natural number i. Prove that if $A_1 - A_n$ and $A_2 + A_{n-1}$ are divisible by 1999, then n is odd.

8.18 (Leningrad Math Olympiad 1988 Grade 10 Elimination Round) The function $F\colon \mathbf{R} \to \mathbf{R}$ is continuous and $F(x) \cdot F(F(x)) = 1$ for all real x. It is known that $F(1000) = 999$. Find $F(500)$.

8.19 (Leningrad Math Olympiad 1990 Grade 11 Elimination Round) A continuous function $f\colon \mathbf{R} \to \mathbf{R}$ satisifes equality $f(x + f(x)) = f(x)$ for all real x. Prove that f is constant.

8.20 (Leningrad Math Olympiad 1991 Grades 9-10 Elimination Round) Does there exist a function $F\colon \mathbf{N} \to \mathbf{N}$ such that for any natural number x,

$$F(F(F(\cdots F(x)\cdots))) = x + 1\,?$$

Here F is applied $F(x)$ times.

8.21 (Leningrad Math Olympiad 1989 Grade 9 Elimination Round) A sequence of real numbers $a_1, a_2, a_3, \ldots$ has the property that $a_{k+1} = (ka_k + 1)/(k - a_k)$ for any natural number k. Prove that this sequence contains infinitely many postitive terms and infinitely many negative terms.

8.22 (Leningrad Math Olympiad 1989 Grade 10 Elimination Round) A sequence of real numbers $a_1, a_2, a_3, \ldots$ has the property that $|a_m + a_n - a_{m+n}| \le 1/(m+n)$ for all m and n. Prove that this sequence is an arithmetic progression.

8.23 (Leningrad Math Olympiad 1991 Grade 11 Elimination Round) The finite sequence $a_1, a_2, a_3, \ldots, a_n$ is called p-balanced if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is $p - \mathit{balanced}$ for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to 0.

8.24 (Int. Math Olympiad 1977) Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if $f(n+1) > f(f(n))$ for each positive integer n, then $f(n) = n$ for each n.

8.25 (Int. Math Olympiad 1976). A sequence $\{u_n\}$ is defined by $u_0 = 2$, $u_1 = 5/2$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for $n = 1, 2, \ldots$. Prove that for positive integers n,

$$[u_n] = 2^{[2^n - (-1)^n]/3}$$

where $[x]$ denotes the greatest integer less than or equal to x.

8.26 (Bratislava Correspondence Seminar, Fall 1999 3rd series): Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ that satisfy: $xf(x) + f(1-x) = x^3 - x$ for all real x.

8.27 (Bratislava Correspondence Seminar, Fall 1999 3rd series): Let $f_1, f_2, f_3, \ldots$ be the elements of the *Fibonacci sequence* (that is, $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all positive integers n). Prove that if $P(x)$ is a a polynomial of degree 998 for which $P(k) = f_k$ for $k = 1000, 1001, \ldots, 1998$, then $P(1999) = f_{1999} - 1$.

8.28 (Bratislava Correspondence Seminar, Fall 1998 3rd series): For a function $f\colon \mathbf{Z} \to \mathbf{R}$, the following statement is true:

$$f(z) = \begin{cases} z - 10 & \text{for } z > 100 \\ f(f(z+11)) & \text{for } z \leq 11 \end{cases}$$

Prove that for all $z \leq 100$, $f(z) = 91$.

8.29 (Bratislava Correspondence Seminar, Fall 1998 3rd series — but I'm sure this problem is not original): $f\colon \mathbf{R} \to \mathbf{R}$ is continuous and $f(f(f(x))) = x$ for all real x. Prove that $f(x) = x$ for all real x.

8.30 (British Math Olympiad 1999). Any positive integer m can be written uniquely in base 3 as a string of 0s, 1s, and 2s (not beginning with a zero). For example:

$$98 = (1 \cdot 81) + (0 \cdot 27) + (1 \cdot 9) + (2 \cdot 3) + (2 \cdot 1) = (10122)_3\,.$$

Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance $c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18$. For any fixed positive integer n, define the sequence (u_r) by:

$$u_1 = n \;\text{ and }\; u_r = c(u_{r-1}) \;\text{ for } r \geq 2$$

Show there is a positive integer r for which u_r is in the set $\{1, 2, 17\}$.

8.31 (British Math Olympiad 1999) Consider all functions f from the positive integers to the positive integers such that:

(i) for each positive integer m there is a unique positive integer n such that $f(n) = m$.

(ii) for each positive integer n, we have either $f(n+1)$ is either $4f(n) - 1$ or $f(n) - 1$.

Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

8.32 (Putnam, 1999, problem A-6) The sequence $(a_n)_{n \geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}\,.$$

Show that, for all n, a_n is an integer multiple of n.

8.33 (Putnam 1990) Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$ and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}\,.$$

The first few terms are 2, 3, 6, 14, 40, 152, 784, 5158, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$ where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

8.34 (Putnam 1980) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?

8.35 (USAMO 1993) Consider functions $f\colon [0,1] \to \mathbf{R}$ which satisfy:

1. $f(x) \geq 0$ for all x in $[0,1]$,
2. $f(1) = 1$,
3. $f(x) + f(y) \leq f(x+y)$ whenever x, y, and $x+y$ are all in $[0,1]$.

Find, with proof, the smallest constant c such that $f(x) \leq cx$ for every function f satisfying the three conditions and every x in $[0,1]$.

8.36 (USAMO 1993) Let a, b be odd positive integers. Define the sequence f_n by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

8.37 (India, 1998) Let N be a positive integer such that $N+1$ is prime. Choose a_i from $\{0,1\}$ for $i = 0, \ldots, N$. Suppose that the a_i are not all equal, and let $f(x)$ be a polynomial such that $f(i) = a_i$ for $i = 0, \ldots, N$. Prove that the degree of $f(x)$ is at least N.