

QUANTUM NUMBERS

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0.1. Algebra of polynomials.

Question. What is a polynomial? In schools we are taught that a polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0, \quad (1)$$

where a_i are real numbers. In particular we can draw the graph of a polynomial, compute values of a polynomial, find derivatives, etc. In contrast to that, in this notes we will adopt a purely algebraic point of view.

Definition. A ring of polynomials is just the set of objects of the form (1), which can be added and multiplied.

We can think of polynomials as of a kind of a game. We have pieces called $1, x, x^2, x^3, \dots$. Let us take 3 pieces “ x^3 ”, 5 pieces “1” and half of the piece x . So, we have a “polynomial” $x^3 + 0.5x + 5$.

Remark: We should not worry to much about polynomials like $\pi x - \sqrt{3}$. Minus just means that we give out some pieces instead of taking and there is nothing wrong with taking π fraction of “ x ”, after all it is just a little more than $3.14 x$ and a little less than $3.15 x$.

Now, we can add polynomials in the usual way: take polynomials p_1 and p_2 and form the polynomial $p_1 + p_2$ just putting all the pieces together. For example, if we had the polynomials $x^3 + 0.5x + 5$ and $2x^2 + 3x + 1$, the result of addition as you may expect would be $x^3 + 2x^2 + 2.5x + 6$.

A more sophisticated operation allowed in our game is multiplication. It works as follows. First let us explain what multiplication by a number means. To multiply a polynomial $x^3 + 0.5x + 5$ by, say, 4 means take 4 copies of it and put everything together, so $4 \times (x^3 + 0.5x + 5) = 4x^3 + 2x + 20$. Note that, in particular, the multiplication by 1 does nothing to a polynomial.

Next, what is multiplication by x^k ? It is just an operation of trading: we say we multiply by x^k and we just trade all pieces of type “ x^n ” to pieces “ x^{n+k} ”. We get $x^k \times (x^3 + 0.5x + 5) = x^{k+3} + 0.5x^{k+1} + 5x^k$.

Finally, if we want to multiply by polynomial which consists of several terms (pieces) we multiply by each piece separately and add the results. Here is an example:

$$(x + a) \times (x - a) = x \times (x - a) + a \times (x - a) = (x^2 - ax) + (ax - a^2) = x^2 - a^2.$$

Thus we obtain our usual operations of addition and multiplication.

Remark: So far it looks that we did not discover much. However, we will see that such a point of view can lead to some interesting mathematics.

Sets of objects where we have two operations: addition and multiplication with properties similar to the ones we described are called rings (in some cases a ring is also called an algebra, in fact we do have an algebra).

The set of pieces $\{1, x, x^2, x^3, \dots\}$ is called a basis of our ring.

Notice multiplying x to itself many times, we generate all the “pieces” x^k . This is why x is called a generator of our ring.

Notice also that our multiplication is commutative, it does not matter in what order we multiply: $p_1(x)p_2(x) = p_2(x)p_1(x)$. Later we will deal with noncommutative rings where this is no longer true.

Let us use a similar approach to differentiation.

Definition: A differentiation $d = \frac{d}{dx}$ is another operation which given a polynomial p produces another polynomial dp . This operation has the properties:

$$d(a_1p_1(x) + a_2p_2(x)) = a_1dp_1(x) + a_2dp_2(x), \quad (2)$$

called linearity (or the sum rule),

$$d(p_1(x)p_2(x)) = (dp_1(x))p_2(x) + p_1(x)(dp_2(x)), \quad (3)$$

called Leibnitz rule (the product rule) and $dx = 1$. Here $p_i(x)$ are polynomials and a_i are real numbers.

Lemma 1. We have $da = 0$, for any real number a , also $dx^n = nx^{n-1}$. More generally, for $k \leq n$, we have

$$d^k(ax^n) = n(n-1)\dots(n-k+1)ax^{n-k} = a\frac{n!}{k!}x^{n-k},$$

and $d^kx^n = 0$ if $k > n$.

Proof: Let us compute the derivative of x^2 .

$$d(x^2) = d(xx) = (dx)x + xdx = x + x = 2x.$$

Exercise: Finish the proof of the lemma.

□

Now we have two different operations: multiplication by x and differentiation d . What if we apply them in different orders?

Lemma 2. $dx - xd = 1$. In other words, for any polynomial $p(x)$, we have $d(xp(x)) - xd(p(x)) = p(x)$.

Exercise: Prove the lemma.

0.2. **Exponentials.** Recall that the number e is defined by the following sum:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Exercise: Prove that e is irrational.

Definition: We define an object called e^x by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

This is an example of an object called a power series. “Power” because it is made of powers of x and “series” for obvious reasons. For us, a power series is not much different

from a polynomial - we just have infinitely many terms (pieces) at the same time. We still can multiply and add power series according to the same rules.

Exercise: Check that $(1-x)(1+x+x^2+x^3+\dots) = 1$, so we can write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \quad (4)$$

Exercise: Prove that $de^x = e^x$.

Consider now the ring of polynomials in two variables x and y . These are objects of the type

$$\sum_{k,l} a_{kl}x^k y^l,$$

which again form a ring (that is we can multiply and add such expressions). Note that we assumed $xy = yx$. It means that we write our variables in any order we like (we wrote y on the right to x). Here is an example: $2 - 3x + 4.2y^2x^3$.

Exercise: Work out the addition, the multiplication, the differentiations with respect to x and y .

The main property of the exponential function is

Lemma 3. $e^{x+y} = e^x e^y$.

Proof: Just multiply everything out and use the Newton binomial formula:

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \dots + \binom{n}{n}y^n, \quad (5)$$

where the binomial coefficients are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Exercise: Work out the details.

□

Our next goal is to compute e^d . Namely, if $p(x)$ is a polynomial, what would be $e^d p(x)$?

Lemma 4. We have $e^d p(x) = p(x+1)$ and, more general, for any number h , we have

$$e^{hd} p(x) = p(x+h).$$

Proof: It is enough to consider the case $p(x) = x^n$ (why?). We have

$$\begin{aligned} e^{hd} x^n &= \left(1 + hd + \frac{h^2 d^2}{2!} + \frac{h^3 d^3}{3!} + \frac{h^4 d^4}{4!} + \dots\right) x^n = \\ &= x^n + nhx^{n-1} + \frac{n(n-1)}{2} h^2 x^{n-2} + \frac{n(n-1)(n-2)}{3!} h^3 x^{n-3} + \dots + h^n = (x+h)^n, \end{aligned}$$

by the binomial formula. (Note that we have only finitely many non-zero terms.) □

Remark: In calculus, the formula $e^{hd}p(x) = p(x+h)$ is called the Taylor expansion formula. It is an important formula, and it includes many others as special cases. For example, if $p(x)$ is a polynomial then as we saw this is just the Newton binomial formula.

If $p(x) = 1/(1-x)$ then the Taylor expansion formula (where x is set to zero) becomes (4) (with x replaced by h) - just the formula for the geometric progression.

Note that in calculus, the Taylor expansion formula does not necessarily hold for all values of x . But in algebraic formal sense it always true.

Exercise: a) For what values of x is the formula (4) true?

b) Let $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$. Use calculus to show that $(e^{hd}f)(x)|_{x \rightarrow 0} = 0$ for all h .

Now let us consider two operation: e^x and e^{hd} .

Lemma 5. $e^{hd}e^x = qe^xe^{hd}$, where $q = e^h$. It means that for any polynomial (even power series) $p(x)$ we have $e^{hd}e^xp(x) = qe^xe^{hd}p(x)$.

Proof:

$$e^{hd}e^xp(x) = e^{x+h}p(x+h) = e^he^xp(x+h) = qe^xe^{hd}p(x).$$

□

Exercise: Let A and B are two letters such that $AB - BA = h$. Prove that $e^Ae^B = qe^Be^A$ by multiplying the corresponding series and moving all A 's to the right and all B 's to the left.

Remark: Our notations are not completely random - h is the standard notation for the Plank constant, and q is traditionally related to the word quantum.

0.3. Gaussian q -numbers. What we do next was known already to Gauss but the deep meaning of these games became clear much later after invention of quantum mechanics and related objects. In fact we really study "a free quantum particle on a line".

Question. What are binomial coefficients?

One possible answer to this question is the following. The binomial coefficients are coefficients in the Newton binomial formula. That is we take to letters x, y , and expand powers of the sum $x+y$ assuming (usually silently!) that $xy = yx$ then we find numbers which are called binomial coefficients. Note also that the first binomial coefficient $\binom{n}{1}$ is actually number n itself.

Consider the ring of polynomials in variables X, Y such that $YX = qXY$ for some real number q . These polynomials are again objects of the type

$$\sum_{k,l} c_{kl} X^k Y^l,$$

where $c_{k,l}$ are some numbers and we add them as usual. However the multiplication changes. Let us consider some examples:

$$(XY^2+Y)(X+XY) = XY^2X+XY^2XY+YX+YXY = q^2X^2Y^2+q^2X^2Y^3+qXY+qXY^2,$$

$$(X+Y)(X-Y) = X^2 - Y^2 + (q-1)XY.$$

Note that if we set $q = 1$ then we get back to good old (classical) multiplication, where $(x - y)(x + y) = x^2 - y^2$.

Definition: The q -binomial coefficients $\binom{n}{k}_q$ are coefficients in the expansion of powers of $(X + Y)^n$,

$$(X + Y)^n = X^n + \binom{n}{1}_q X^{n-1}Y + \binom{n}{2}_q X^{n-2}Y^2 + \binom{n}{3}_q X^{n-3}Y^3 + \dots + \binom{n}{n}_q Y^n.$$

and the quantum number $[n]_q$ is the first q -binomial coefficient $[n]_q = \binom{n}{1}_q$.

First we derive the q -analogs of the Pascal identities.

Lemma 6.

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + q^{n-k} \binom{n-1}{k-1}, \\ \binom{n}{k} &= q^k \binom{n-1}{k} + \binom{n-1}{k-1}. \end{aligned}$$

Proof: Multiply $(X + Y)^{n-1}$ by $(X + Y)$ from the left. Then the term $X^{n-k}Y^k$ in the result is obtained multiplying the term $X^{n-k-1}Y^k$ in $(X + Y)^{n-1}$ by X and by multiplying the term $X^{n-k}Y^{k-1}$ in $(X + Y)^{n-1}$ by Y . Note that in the second case we have to swing Y to the right through X^{n-k} which will give a factor q^{n-k} . Now we obtain the first equality by comparing the coefficients in front of $X^{n-k}Y^k$.

The second equality is proved similarly multiplying $(X + Y)^{n-1}$ by $(X + Y)$ from the right. \square

Now we are ready to compute the q -binomial coefficients explicitly.

Lemma 7. *The q -binomial coefficients are polynomials in q with nonnegative integer coefficients, which are equal to usual binomial coefficients when $q = 1$. Explicitly we have*

$$[n]_q = 1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q},$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1 - q^n)(1 - q^{n-1}) \dots (1 - q^{n-k+1})}{(1 - q)(1 - q^2) \dots (1 - q^k)},$$

where $[n]_q! = [1]_q [2]_q \dots [n]_q$.

Proof: Induction on n .

Exercise: Fill out the details.

\square

Many identities with binomial coefficients can be “deformed” (“quantized”) to the identities for the q -binomial coefficients. The Pascal identity is one example.

Another example is the Chu-Wandermond identity.

Exercise: Prove that

$$\binom{m+n}{k}_q = \sum_{i=0}^k q^{(m-i)(n-i)} \binom{m}{i}_q \binom{n}{k-i}_q.$$

0.4. **The case $q^l = 1$.** Let $q^l = 1$ and $q^s \neq 1$ for $s = 1, \dots, l-1$. Then we have $[n]_q = [n+l]_q$. And there are only l different quantum numbers, in particular $[l]_q = 0$. Note that the quantum numbers are still polynomials in q . The only difference is that we can now reduce high powers of q to the smaller ones using the relation $q^l = 1$.

The situation is somewhat similar to the reduction of all integers to reminders modulo l . In fact this similarity is very deep. Here we show some examples.

Exercise: Let l be a prime number. Let m_1, m_2 , be natural numbers. Devide m_1, m_2 by l with a reminder: $m_1 = k_1l + r_1, m_2 = k_2l + r_2$, where $0 \leq r_1, r_2 < l$. Prove that

$$(x+y)^l = x^l + y^l \pmod{l},$$

$$\binom{m_2}{m_1} = \binom{k_2}{k_1} \binom{r_2}{r_1} \pmod{l}.$$

We have the following q -analog of this identities.

Exercise: Let $q^l = 1$ and $q^s \neq 1$ for $s = 1, \dots, l-1$. Let m_1, m_2 , be natural numbers. Devide m_1, m_2 by l with a reminder: $m_1 = k_1l + r_1, m_2 = k_2l + r_2$, where $0 \leq r_1, r_2 < l$. Prove that (we recall the relations $YX = qXY$).

$$(X+Y)^l = X^l + Y^l,$$

$$\binom{m_2}{m_1}_q = \binom{k_2}{k_1}_q \binom{r_2}{r_1}_q.$$

0.5. **q-exponent and q-differentiation.** Many objects have their quantum versions. We describe here the q -versions of the exponential function and the differentiation.

Recall that the usual derivative d is a linear map such that $dx^k = kx^{k-1}$. As before, linear means that we differentiate term by term, see (2) above.

Definition: Define q -derivative D_q of as a linear map such that $D_q X^k = [k]_q X^{k-1}$.

Remark: As always at $q = 1$ we recover the classical definition of the derivative.

It turns out that there is a property similar to the Leibnitz rule (3).

Lemma 8.

$$D_q(p_1(X)p_2(X)) = (D_q f(X))g(X) + f(qX)(D_q g(X)) = (D_q f(X))g(qX) + f(X)(D_q g(X)).$$

Exercise: Proof the lemma.

The calculus type definition of the usual derivative involves taking a certain limit: one has to consider infinitely small changes of the variable. A nice thing is that in the q -case this definition becomes easy.

Lemma 9.

$$D_q p(X) = \frac{p(X) - p(qX)}{X - qX}.$$

Exercise: Prove the lemma and observe what happens in the limit $q \rightarrow 1$.

In the same spirit, let us define the q -exponential.

Definition: The q -exponential $E_q(X)$ is defined by the formula

$$E_q(X) = 1 + X + \frac{X^2}{[2]_q!} + \frac{X^3}{[3]_q!} + \dots$$

Exercise: Prove that $D_q E_q(X) = E_q(X)$.

Exercise: Let $YX = qXY$. Prove that $E_q(X+Y) = E_q(X)E_q(Y)$. Note that we cannot switch factors in the RHS.

Exercise: Let $YX = qXY$. Prove the q -version of the Taylor expansion formula:

$$p(X+Y) = E_q(X D_q^Y) p(Y),$$

where D_q^Y is the q -differentiation with respect to Y variable.

0.6. More Exercises.

1. Prove the chain rule formula: $d(p_1(p_2(x))) = dp_1(p_2(x)) dp_2(x)$.

2. Use calculus show that

$$(e^{hd} \sin(x))|_{x \rightarrow 0} = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \dots, \quad (6)$$

$$(e^{hd} \cos(x))|_{x \rightarrow 0} = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \dots \quad (7)$$

3. Denote $\sin(h)$ and $\cos(h)$ to be the right hand side power series in (6) and (7) respectively. Show that $\sin^2(h) + \cos^2(h) = 1$. Show that $dsin(x) = cos(x)$, $dcos(x) = -sin(x)$. Let i be a symbol, such that $i^2 = -1$. Show that $e^{ih} = \cos(h) + i \sin(h)$.

4. Prove the formula

$$\prod_{k=0}^{n-1} (1 + xq^k) = \sum_{k=0}^n q^{\frac{k(k-1)}{2}} \binom{n}{k}_q x^k.$$

5. Consider the area A under the graph of x^k on the segment $[0, 1]$. Let us chose $q < 1$ and divide $[0, 1]$ in infinitely many segments $[q^{n+1}, q^n]$, $n = 0, 1, 2, \dots$. Then our area A can be approximated by the sum $s(q)$:

$$A \simeq s(q) := \sum_{n=0}^{\infty} (q^n - q^{n+1}) q^{nk} = \frac{1-q}{1-q^{k+1}} = \frac{1}{[k+1]_q}.$$

Prove that $A = \lim_{q \rightarrow 1} s(q)$.

6. Prove that

$$\sum_{k=0}^n (-1)^k \binom{n}{k}_q = (1-q)(1-q^2) \dots (1-q^{n-1}),$$

if n is even and 0 if n is odd.

7. Prove that $D_q X - qX D_q = 1$.