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Problem Solving for "Beginners": Hints and Solutions

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Here are some hints, full solutions, and solution sketches to the problems you got on Feb. 13, 2000. Some of these problems were quite difficult. Remember: you don't have to solve every problem, but you should cultivate a taste for (and stamina for) *investigating* problems.

1 If there are n people in the room, after one minute, there will be either $n+1$ or $n-2$ people. The difference between these two possible outcomes is 3. Continuing for longer times, we see that

> *At any fixed time t, all the possible values for the population of the room differ from one another by multiples of 3.*

In 3^{1999} minutes, then, one possible population of the room is just 3^{1999} people (assuming that one person entered each time). This is a multiple of 3, so *all* the possible populations for the room have to also be multiples of 3. Therefore $3^{1000}+2$ will not be a valid population.

2 This is yet another application of the Handshake Lemma, which says that if you add the number of handshakes each person in a group shakes, the sum will be even (since you are double-counting). Let $d_0, d_1, d_2, \ldots, d_n$ be the number of doors in room i, where we include "outside" as room 0 (having as its doors each of the doors in the house that open to the outside). Then by the same reasoning as in 3.4.7, the sum $d_0 + d_1 + d_2 + \ldots + d_n$ must be even. By hypothesis, $d_1, d_2, \ldots, d_n$ are all even, which forces d_0 to be even as well.

3 Experiment, and you will guess that only perfect squares stay closed. This is due to the parity: the number of divisors of n is odd if and only if n is a perfect square.

4 Consider the *sum* of the terms. We have

$$\begin{aligned}
&(a_1-1)+(a_2-2)+\cdots+(a_n-n)\\
&\quad=(a_1+a_2+\cdots+a_n)-(1+2+\cdots+n)\\
&\quad=(1+2+\cdots+n)-(1+2+\cdots+n)\\
&\quad=0,
\end{aligned}$$

so the sum is an invariant; it is equal to zero no matter what the arrangement. A sum of an odd number of integers which equals zero (an even number) must contain at least one even number!

5 Let us indicate the population at any time by an ordered triple (x, y, z). Without loss of generality, consider an X-Z collision. The new population right, etc. The important thing about this sequence is that P_7 ends up on the left side, along with P_1. Therefore L cannot pass through the interior of segment P_1P_7, a contradiction.

7 The answer is no. To see why, assume that there is a tiling, and we shall look for a contradiction. Color the squares of 66×62 rectangle with 12 colors in a cyclic "diagonal" pattern as follows (we are assuming that the height is 66 and the width is 62):

1	12	11	$\cdots$	1	12
2	1	12	$\cdots$	2	1
3	2	1	$\cdots$	3	2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
5	4	3	$\cdots$	5	4
6	5	4	$\cdots$	6	5

and then look at

60×60	60×2
6×60	6×2

.

8 It is easy to verify that if (x, y) is a legal point, then $y - x$ will be a multiple of 11. Since $1999 - 3$ is not a multiple of 11, the answer is no.

9 Notice that $uv + u + v + 1 = (u + 1)(v + 1)$, so that u, v are replaced with $(u+1)(v+1)-1$. Therefore, if the sequence contains the values $a_1, a_2, \ldots, a_n$, the quantity

$$(a_1 + 1)(a_2 + 1)(a_3 + 1) \cdots (a_n + 1)$$

is invariant! Hence the final number will equal $100! - 1$, no matter what the choices are.

10 For example, if $n = 6$ and the starting sequence was 362154, the cards evolve as follows:

$$\begin{aligned} 362154 &\rightarrow 263154 \rightarrow 623154 \rightarrow 451326 \rightarrow 315426 \\ &\rightarrow 513426 \rightarrow 243156 \rightarrow 423156 \rightarrow 132456. \end{aligned}$$

It would be nice if the number of the card in the 1st place decreased monotonically, but it didn't (the sequence was $3, 2, 6, 4, 3, 5, 2, 4, 1$). Nevertheless, it is worth thinking about this sequence. We shall make use of a very simple, but important, general principle:

> *If there are only finitely many states as something evolves, either a state will repeat, or the evolution will eventually halt.*

In our case, either the sequence of 1st-place numbers repeats (since there are only finitely many), or eventually the 1st-place number will be 1 (and then the evolution halts). We would like to prove the latter. How do we exclude the possibility of repeats? After all, in our example, there were plenty of repeats!

Once again, the extreme principle saves the day. Since there are only finitely many possibilities as a sequence evolves, there exists a *largest* 1st-place value that ever occurs, which we will call L_1 (in the example above, $L_1 = 6$). So, at some point in the evolution of the sequence, the 1st-place number is L_1, and thereafter, no 1st-place number is ever larger than L_1. What happens immediately after L_1 occurs in the 1st place? We reverse the first L_1 cards, so L_1 appears in the L_1th place. We know that the 1st-place card can never be larger than L_1, but can it ever again equal L_1? The answer is no; as long as the 1st place value is less than L_1, the reversals will never touch the card in the L_1th place. We will never reverse more than the first L_1 cards (by the maximality of L_1), so the only way to get the card numbered L_1 to move at all would be if we reversed exactly L_1 places. But that would mean that the 1st-place and L_1th-place cards both had the value L_1, which is impossible.

That was the crux move. We now look at all the 1st-place values that occur after L_1 appeared in the 1st place. These must be *strictly* less than L_1. Call the maximum of these values L_2. After L_2 appears in 1st place, all subsequent 1st-place values will be strictly less than L_2 by exactly the same argument as before.

Thus we can define a *strictly decreasing* sequence of maximum 1st-place values. Eventually, this sequence must hit 1, and we are done!

11 Let us call any set $\{x_1, x_2, \ldots, x_{23}\}$ of integers “balanced” if it has the property that no matter which of the x_i is chosen for the referee, then one can decompose the remaining 22 numbers into two sets of 11 which have equal sums. Clearly if a set is balanced and we multiply or divide each element by the same number, it will still be balanced. Likewise, if a set is balanced and we add or subtract the same number to each element, it will still be balanced.

Now, let us suppose we have a balanced set $\{x_1, x_2, \ldots, x_{23}\}$ of positive integers. Let S be the sum of the 23 elements. If we pick x_1 as referee, then we know that $S - x_1$ must be *even*, since the remaining 22 elements can be partitioned into two sets with equal integer sums. By the same reasoning, $S - x_2, S - x_3, \ldots, S - x_{23}$ are all even. Therefore, if the set of integers is balanced, then all the elements $x_1, x_2, \ldots, x_{23}$ are the same parity (i.e., all are even, or all are odd).

Now consider our balanced set $\{x_1, x_2, \ldots, x_{23}\}$ of positive integers. We wish to show that all elements are equal. Let a be the minimum value of the elements. If we define $b_i = x_i - a$ for $i = 1, 2, \ldots, 23$, then the new set $\{b_1, b_2, \ldots, b_{23}\}$ will also be a balanced set of *nonnegative* integers. Some of the elements will be zero, and perhaps some are not. We would like to prove that they are all zero. Since some of the elements are zero, and zero is even, then all of the elements must be even. Consequently we can form a new set $\{c_1, c_2, \ldots, c_{23}\}$, where $c_i = b_i/2$ for $i = 1, 2, \ldots, 23$. But this set also has some zero elements, hence all of its elements are even, hence we can divide them all by 2 and get yet another balanced set of nonnegative integers. We can do this forever! The only integer which one can divide by 2 endlessly and still get an even integer as a result is zero. We conclude that the elements of $\{b_1, b_2, \ldots, b_{23}\}$ are all zero, i.e., the elements of $\{x_1, x_2, \ldots, x_{23}\}$ are all equal.

12 Consider the general problem of n marbles $m_0, m_1, \ldots, m_{n-1}$ with arbitrary starting locations. Each marble has a "ghost path," the path it would travel if it did not bounce off its neighbors but instead passed through them. Whenever the marbles bounce, the actual path of a marble coincides with another marble's ghost path. After one minute has passed, each ghost path has returned to the original positions of each marble. Hence after one minute, the actual locations of the marbles are a permutation of the original positions. Moreover, this permutation must be a cyclic permutation, since the marbles cannot pass through one one another.

We claim that the permutation takes m_0 to m_d, where d is the "counterclockwise excess," i.e. the difference modulo n between the number of counterclockwise marbles and the number of clockwise marbles.

To see this, let $v_i(t)$ be the *velocity* function for marble m_i, where the velocities of $+1, -1$ denote counterclockwise and clockwise motion, respectively. Notice that for any time t,

$$\sum_{i=0}^{n-1} v_i(t) = d,$$

since the number of clockwise and the number of counterclockwise marbles never changes (even when marbles collide). There will be finitely many bounces, and in any time interval between bounces, each velocity function is a constant. Let $t_1, t_2, \ldots, t_k$ be time values inside each interval, and let each each interval have length ℓ_i. For each marble m_i, denote the net counterclockwise distance traveled from $t = 0$ to $t = 1$ by

$$s_i = v_i(t_1)\ell_1 + v_i(t_2)\ell_2 + \cdots + v_i(t_k)\ell_k.$$

Summing this over all marbles, we get

$$\sum_{i=0}^{n-1} s_i = d(\ell_1 + \ell_2 + \cdots + \ell_k) = d \cdot 1 = d.$$

The only cyclic permutation associated with this sum of net distance traveled is the one which takes m_0 to m_d.

13 A very sketchy hint: orient the large rectangle so that one corner is a lattice point, and then consider the parity of the number of corner lattice points.

14 The idea is that triangular packing (so the centers of three circles are vertices of an equilateral triangle) is a more efficient use of space, however, to do this, you waste some space at the beginning of the rectangle. So you need a long rectangle to catch up.

15 Let us work out the first few terms of the product. We get

$$(1+x^3)(1+2x^9)(1+3x^{27})(1+4x^{81})\cdots =$$

$$= 1 + x^3 + 2x^9 + 2x^{12} + 3x^{27} + 3x^{30} + 6x^{36} + 6x^{39} + 4x^{81} + \cdots.$$

What are the (positive) exponents k_i? All integers of the form $3^{u_1} + 3^{u_2} + \cdots + 3^{u_r}$, where the integers u_j satisfy $1 \le u_1 < u_2 < \cdots < u_r$. In other words, they will be numbers which, when written in *base-3*, only contain ones and zeros and end with a zero. In order, the first few exponents are (written in base-3) are

$$10, 100, 110, 1000, 1010, 1100, 1110, 10000, \ldots.$$

Of course, these numbers are just the *base-2* representations of the sequence $2, 4, 6, 8, \ldots$. In particular, to figure out k_{1996}, we just write $2 \cdot 1996 = 3992$ in base-2:

$$3992 = 2048 + 1024 + 512 + 256 + 128 + 16 + 8,$$

so the base-2 representation of 3994 is 111110011000, and k_{1996} is equal to 111110011000 (base-3).

In other words,

$$k_{1996} = 3^3 + 3^4 + 3^7 + 3^8 + 3^9 + 3^{10} + 3^{11}.$$

16 See *Concrete Mathematics* by Graham, Knuth and Patashnik for a very nice discussion of this and related problems.

17 A bit of experimentation convinces us that if $n = 3$, the total private area is also equal to the total area of one planet. Playing around with larger n suggests the same result. We conjecture that the total private area is always equal exactly to the area of one planet, no matter how the planets are situated. It appears to be a nasty problem in solid geometry, but must it be? The notions of "private" and "public" seem to be linked with a sort of duality; perhaps the problem is really not geometric, but *logical*. We need some "notation." Let us assume that there is a universal coordinate system, such as longitude and latitude, so that we can refer to the "same" location on any planet. For example, if the planets were little balls floating in a room, the location "north pole" would mean the point on a planet which was closest to the ceiling.

Given such a universal coordinate system, what can we say about a planet P which has a private point at location x? Without loss of generality, let x be at the "north pole." Clearly, the centers of all the other planets must lie on the south side of the P's "equatorial" plane. But that renders the north poles of these planets public, for their north poles are visible from a point in the southern hemisphere of P (or from the southern hemisphere of an planet that lies between). In other words, we have shown pretty easily that

> *If location x is private on one planet, it is public on all the other planets.*

After this nice discovery, the penultimate step is clear: to prove that

> *Given any location x, it must be private on some planet.*

We leave this as an exercise (problem?) for you!

18 Let R be the interior of the rectangle with vertices $(0,0), (b,0), (b,a), (0,a)$. The line $y = ax/b$ intersects no lattice points in R (it passes through $(0,0)$ and (b,a), but these points are not included in R, and there are no other lattice points on the line, since a and b have no common divisors). Observe that $\lfloor ai/b \rfloor$ is just the number of lattice points that lie below this line in R for $x = i$. Thus the left-hand sum is just the number of lattice points lying below the line in R . By similar reasoning, the right-hand sum is equal to the number of lattice points lying above the line. The common value must equal one-half of the total number of lattice points, which is of course $(a-1)(b-1)$. Observe that at least one of a and b must be odd, for otherwise the two numbers would share a common divisor (namely, 2). Consequently $(a-1)(b-1)$ is even and can be divided by 2.

19 Assume that the values are not all equal. Let $a > 0$ be the smallest value on the board. There must be a square containing a which is adjacent (WLOG) on the east by a square containing the value b which is strictly greater than a. But then a is equal to the average of 4 numbers, none less than a, one of which is strictly greater than a. This is a contradiction.

20 This is very similar to the problem above.

21 The coin with smallest diameter cannot be tangent to more than 5 others.

22 Certainly when n is even, it is not true: Just imagine a set of pairs of people standing a few inches apart, with each pair quite far from every other pair. Now, if n is odd, first eliminate all pairs as in the above case, where two people end up shooting each other. Since n is odd, some dry people remain. Now consider the person whose nearest neighbor is maximal (there may be ties). This person will stay dry, since the only way that he could get shot is if someone else is as close to him as he is to his nearest neighbor. But that contradicts the fact that for each person, the distances to the others are different.

23 $g(n) = 2^r$, where r is the number of 1's in the base-2 representation of n. This can be proven with induction, as well as many other methods.

24 Let there be n people. Each person is seated a distance d from his or her correct place, where $0 < d < n$ is measured counterclockwise. There are n people, but $n-1$ different values of d. Hence at least two people share the same distance d.

25 The bug should travel along two line segments: first from $(7, 11)$ to $O = (0, 0)$, and then from O to $(-17, -3)$. This is a consequence of the following principle: the bug must avoid quandrant II completely, even though a straight line path from $(7, 11)$ to $(-17, -3)$ goes through quadrant II.

To see why this is true, let a and b be arbitrary positive numbers, and consider a path starting at $A = (0, a)$ and ending at $B = (-b, 0)$. Certainly the quickest route *within quadrant II* is the line segment $\overline{AB}$, and the length of this path is $\sqrt{a^2 + b^2}$. Now consider the alternate route $\overline{AO}$ followed by $\overline{OB}$. This path lies outside quadrant II (since quadrant II does not include the x- or y-axes) and has total length $a + b$. Compare these two lengths. By the arithmetic-geometric mean inequality, we have $a^2 + b^2 \geq 2ab$, which implies that $2a^2 + 2b^2 \geq a^2 + 2ab + b^2 = (a + b)^2$. Hence

$$a + b \leq \sqrt{2}\sqrt{a^2 + b^2}.$$

We conclude that as long as the speed in quadrant II is less than $\frac{1}{\sqrt{2}}$, then any path from A to B that passes through quadrant II will take more time than the shortest non-quadrant-II path (along the y- and x-axes). Since $\frac{1}{2} < \frac{1}{\sqrt{2}}$, our bug will save time by avoiding quadrant II.

26 This problem is rather tricky unless we start by considering the 2-dimensional case. A bit of playing around convinces us that 5 is the magic number: If 5 lattices points are chosen in the plane (all distinct, of course), then one of the line segments joining two of these points will have a lattice point in the interior. The key ideas are parity and pigeonhole. There are only four distinct parity types for lattice points: (odd, odd), (odd, even), (even, odd), and (even, even). Hence among any 5 distinct lattice points, two must be of the same parity type, *which means that the midpoint of the line segment joining them is a lattice point!* The argument adapts easily to 3-dimensions.

27 Let triangle ABC have the largest area among all triangles whose vertices are taken from the given set of points. Let $[ABC]$ denote the area of triangle ABC. Then $[ABC] \leq 1$. Let triangle LMN be the triangle whose **medial triangle** is ABC. (In other words, A, B, C are the midpoints of the sides of triangle LMN. See figure.)

medial$_{t}riangle.epsscaled$500 Then $[LMN] = 4[ABC] \leq 4$. We claim that

the set of points must lie on the boundary or in the interior of LMN. Suppose a point P lies outside LMN. Then we can connect P with two of the vertices of ABC forming a triangle with larger area than ABC, contradicting the maximality of $[ABC]$.

28 We will show that no palindrome can exist by contradiction. Assume that the concatenation of the numbers from 1 to n was the palindrome

$$P := 1234567891011 \cdots 4321.$$

Consider the longest run of consecutive zeros in P; note that this exists, since n is surely greater than 10. There may be several runs of consecutive zeros that are all equally long; pick the last (rightmost) one. Observe that immediately to the left of this string is a single digit, and this digit plus the zeros forms one of the numbers from 1 to n. For concreteness, suppose that the longest string of zeros was 0000. Then the rightmost such string obviously consists of the last digits of one of the numbers from 1 to n, not the middle of one, and doesn't straddle two (for example, if the number was, say, 400005, then the number 400000 would have appeared to the left of it, contradicting the fact that 0000 is longest string of zeros. Likewise, the number that ends with 0000 had to start with a single digit, for if, say, the number was 7310000 then there would have been the number 7000000 to the left of it.

So, let us suppose that the rightmost string of 0000 is the last digits of the number 70000. Then, writing the predecessor and successor numbers, these four zeros are embedded in the string 699997000070001 Assume also, that there is at least one other string of 0000 in P. Since P is a palindrome, the first 0000 must be embedded in the string 100070000799996. But that makes no sense, since the first time 0000 appears is as the last digits of the number 10000.

So the only remaining possibility is that there is only one 0000 string, which by necessity is at the exact center of P and is the last four digits of the number 10000. Writing the predecessor and successor, and letting "|" mark the exact midpoint of P, we must have the following string at the center:

$$\cdots 9999100|0010001 \cdots$$

But this isn't symmetrical ($9 \neq 0$), achieving our contradiction.

29 Consider the shortest path joining 1 with n^2, where path means a walk along adjacent squares. The worst case scenario is that the path has length n (if 1 and n^2 are at opposite diagonal corners). In any event, the members of the path will be at most n distinct numbers between 1 and n^2, inclusive. If their successive differences were all less than or equal to n, that means there are $n-1$ successive differences which bridge the gap from 1 to n^2. Since $n^2 - 1 = (n-1)(n+1)$, the largest difference must be at least $n+1$.

30 A very brief hint: show that eventually, the sequence will form a chain where each element will divide the next (when arranged in order). Moreover, the least element and the greatest element of this chain are respectively the greatest common divisor and least common multiple of all the original numbers.

31 Hint: look at diagonals.