Mass Point Geometry (Barycentric Coordinates)

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Berkeley Math Circle January 9, 2000

1 History and Sources

My original intention, when I mentioned this as possible topic was to just show a couple of examples of this technique along with my talk on Archimedes and the Arbelos (January 16, 2000). The words "Mass Point Geometry" were unfamiliar to Zvesda, so I mentioned "Barycentric Coordinates" to give her a notion of what was involved. That is how "Barycentric Coordinates" became part of the title of this talk and how I ended up having two talks this month instead of one.

Mass points were first used by Augustus Ferdinand Möbius in 1827. They didn't catch on right away. Cauchy was quite critical of his methods and even Gauss in 1843 confessed that he found the new ideas of Möbius difficult. This is found in little mathematical note by Dan Pedoe in Mathematics Magazine [**1**]. I first encountered the idea about 25 years ago in a math workshop session entitled "Teeter-totter Geometry" given by Brother Raphael from Saint Mary's College. He apparently always taught one course using only original sources, and that year he was reading Archimedes with his students. It was Archimedes' "principle of the lever" that he used that day to show how mass points could be used to make deductions about triangles. For a very readable account of the assumptions Archimedes makes about balancing masses and locating the center of gravity, I recommend the new book *Archimedes: What Did He Do Besides Cry Eureka?*[**2**] written by Sherman Stein of U.C.Davis.

My next encounter with mass points was in the form of an offer about twenty years ago from Bill Medigovich, who was then teaching at Redwood High School and helping Lyle Fisher coordinate the annual Brother Brousseau Problem Solving and Mathematics Competition. He offered to come to a math club and present the topic of *Mass Points*, if the students would commit to several sessions. I was never able to get my students organized enough, so we missed out on his wonderful presentation. Many years later I asked him for any references he had on the subject and he sent me a packet of the 30 papers [**3**] he used for his presentation. I also found the topic discussed in the appendix of *The New York City Contest Problem Book 1975-1984* [**4**] with a further reference to an article *The Center of Mass and Affine Geometry* [**5**] written by Melvin Hausner in 1962. Recently, Dover Publications reissued a book published by Hausner [**6**] in 1965 that comprised a one year course for high school teachers of mathematics at New York University. The first chapter is devoted to *Center of Mass*, which forms the basis for the entire book. In the preface he credits Professor Jacob T. Schwartz, an eminent mathematician at the Courant Institute of Mathematical Sciences, "who outlined the entire course in five minutes". While we are bringing out big names let me mention Jean Dieudonné, the world famous French mathematician, who went on record saying "Away with the triangle". He wrote a textbook in the 60's for high schools in France which introduces the geometry in the plane and Euclidean space via linear algebra. The axioms are the axioms in the definition of a vector space over a field and no diagrams are given in the book. I looked at the book fifteen years ago and found it very interesting, but I cannot imagine it being used in public schools in the United States. The reason for bringing up Dieudonné at this point is another of his inflammatory comments, "Who ever uses barycentric coordinates?", and the response by Dan Pedoe is to by found in an article by Pedoe entitled *Thinking Geometrically* [**7**].

As I was preparing for this talk, I was going through old issues of *Crux Mathematicorum* and found a key paper on the this subject, *Mass Points* [**8**], that was originally written for the NYC Senior "A" Mathletes. The authors are Harry Sitomer and Steven R. Conrad. The latter author may be familiar to you as the creator of the problems for the past 25 years used in the California Mathematics League as well as all the other affiliated math leagues around the country. I will be using this paper and most of their examples as my main guide for this talk.

2 Definitions and Postulates

Definitions:

1. A mass point is a pair, (n, P) consisting of a positive number n, the weight, and a point P. It will be written as nP for convenience.

2. $nP = mQ$ if and only if $n = m$ and $P = Q$. (Usual equality for ordered pairs)

3. $nP + mQ = (n + m)R$ where R is on $\overline{PQ}$ and $PR : RQ = m : n$. (A weight of n at P and a weight of m at Q will balance iff the fulcrum is place at R since $n(PR) = m(RQ)$.

Postulates:

1. (Closure) Addition produces a unique sum. (There is only one center of mass.)

2. (Commutativity) $nP + mQ = mQ + nP$. (Just view the "teeter-totter" from the other side.)

3. (Associativity) $nP + (mQ + kR) = (nP + mQ) + kR = nP + mQ + kR$. (This sum is called the *center of mass* or *centroid* of the system. The propery is equivalent to the theorem of Menelaus.)

4. (Scalar multiplication) $m(nP) = (mn)P = mnP$.

5. (Idempotent) $nP + mP = (n + m)P$

6. (Homogeneity) $k(nP + mQ) = knP + kmQ$.

7. (Subtraction) If $n > m$ then $nP = mQ + xX$ may be solved for the unknown mass point xX. Namely, $xX = (n - m)R$ where P on $\overline{RQ}$ and $RP : PQ = m : (n - m)$.

3 Examples

Most of the problems here are from the article by Sitomer and Conrad [**8**].

Basics

1. If G is on $\overline{BY}$ then $3B+4Y=xG$. What is x? What is $BG:GY$?

2. If G is on $\overline{BY}$ then $7B+xY=9G$. What is x? What is $BG:GY$?

3. In $\triangle ABC$, D is the midpoint of $\overline{BC}$ and E is the trisection point of $\overline{AC}$ nearer$A$. Let $G=\overline{BE}\cap\overline{AD}$. Find $AG:GD$ and $BG:GE$.
   Solution: Draw the figure! Assign weight 2 to A and weight 1 to each of B and C. Then $2A+1B=3E$ and $1B+1C=2D$. Note that the center of mass of the system is $2A+1B+1C=3E+1C=2A+2D=4G$. From this we can see that $AG:GD=2:2=1:1$ and $BG:GE=3:1$.

4. (**East Bay Mathletes April 1999**) In $\triangle ABC$, D is on $\overline{AB}$ and E is the on $\overline{BC}$. Let $F=\overline{AE}\cap\overline{CD}$. $AD=3$, $DB=2$, $BE=3$ and EC=4 Find $EF:FA$ in lowest terms.

5. Show that the medians of a triangle are concurrent and the point of concurrency divides each median in a ratio of 2:1. (Hint: Assign a weight of 1 to each vertex.) How does this show that the six regions all have the same area?

6. (**Varignon's Theorem (1654-1722)**) If the midpoints of consecutive sides of a quadrilateral are connected, the resulting quadrilateral is a parallelogram. (Hint: Assign weight 1 to each vertex of the original quadrilateral.)

7. In quadrilateral $ABCD$, E, F, G, and H are the trisection points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ nearer A, C, C, A, respectively. Let $\overline{EG}\cap\overline{FH}=K$. Show that $EFGH$ is a parallelogram.

8. Generalize the previous problems for E, F, G, and H divide the sides in a ratio of $m:n$.

Angle Bisectors, Nonconcurrency, Mass Points in Space

1. In $\triangle ABC$, $AB=8$, $BC=6$ and $CA=7$. Let P be the incenter of the triangle and D, E, F be the intersection points of the angle bisectors in side $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Find $AP:PD$, $BP:PE$, and $CP:PF$. (Hint: Assign weight 6 to A, weight 7 to B and weight 8 to C.)

2. Solve the previous problem using $AB=c$, $BC=a$ and $CA=b$.

3. Use the previous problem to prove, as assumed in the previous two problems, that the angle bisectors of the angles of a triangle are concurrent.

4. In $\triangle ABC$, D, E, and F are the trisection points of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ nearer A,B,C, respectively. Let $\overline{BF}\cap\overline{AE}=J$. Show that $BJ:JF=3:4$ and $AJ:JE=6:1$.

5. In the previous problem, let $\overline{CD} \cap \overline{AE} = K$ and $\overline{CD} \cap \overline{BF} = L$. Use the previous problem to show that $DK : KL : LC = 1 : 3 : 3 = EJ : JK : KA = FL : LJ : JB$.

6. Let $ABCD$ be a tetrahedron (triangular pyramid). Assume the same definitions and properties of addition of mass points in space as for in the plane. Assign weights of 1 to each of the vertices. Let G be the point in $\triangle ABC$ such that $1A+1B+1C = 3G$. Then G is the center of mass for $\triangle ABC$. Let F be the point on $\overline{DG}$ such that $1D+3G = F$. F is the center of mass of the tetrahedron. What is the ratio of DF to FG?

7. Show that the four segments from the vertices to centroids of the opposite faces are concurrent at the point F of the previous problem.

8. In tetrahedron $ABCD$, let E be in $\overline{AB}$ such that $AE : EB = 1 : 2$, let H be in $\overline{BC}$ such that $BH : HC = 1 : 2$, and let $\overline{AH} \cap \overline{CE} = K$. Let M be the midpoint of $\overline{DK}$ and let ray HM intersect $\overline{AD}$ in L. Show that $AL : LD = 7 : 4$.

9. Show that the three segments joining the midpoints of opposite edges of a tetrahedron bisect each other. (Opposite edges have no vertex in common.)

10. Let $P - ABCD$ be a pyramid on convex base $ABCD$ with E, F, G, and H the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Let E', F', G', and H', be the respective centroids of $\triangle$'s PCD, PDA, PAB, and PBC. Show that $\overline{EE'}, \overline{FF'}, \overline{GG'}, \overline{HH'}$ are concurrent in a point K which divides each of the latter segments in a ratio of 2:3.

Splitting Masses, Altitudes, Ceva and Menelaus

1. Splitting mass points using $mP+nP = (m+n)P$ is useful when dealing with transversals. In $\triangle ABC$, let E be in $\overline{AC}$ such that $AE : EC = 1 : 2$, let F be in $\overline{BC}$ such that $BF : FC = 2 : 1$, and let G be in $\overline{EF}$ such that $EG : GF = 1 : 2$. Finally, let ray CG intersect $\overline{AB}$ in D. Find $CG : GD$ and $AD : DB$.
   Solution: Draw the figure! Assign weight 2 to C and weight 1 to B so that $2C + 1B = 3F$. It is now necessary to have weight 6 at E to "balance" $\overline{EF}$. Since $1C + 2A = 3E$, we have $2C + 4A = 6E$, so assign *another* weight 2 to C for a total weight of 4 at C and assign a weight of 4 to A. Then $4A + 1B = 5D$. Now the ratios can be read directly from the figure. $CG : GD = 5 : 4$ and $AD : DB = 1 : 4$.

2. In the previous problem, $AE = EC$, $BF : FC = 1 : 2$, and $EG : GF = 2 : 3$. Show that $CG : GD = 17 : 13$ and $AD : DB = 8 : 9$.

3. In the previous problem, let $\overline{CD}$ be a median, let $AE : EC = x : 1$ and $BF : FC = y : 1$. Show that $CG : GD = 2 : (x + y)$ and $EG : GF = (y + 1) : (x + 1)$.

4. For an altitude, say $\overline{AD}$ in $\triangle ABC$, note that $CD \cot B = DB \cot C$. Therefore, assign weights *proportional* to $\cot B$ and $\cot C$ to C and B, respectively. Let $\angle B = 45°$, $\angle C = 60°$, and let the angle bisector of $\angle B$ intersect $\overline{AD}$ in E and $\overline{AC}$ in F. Show that $AE : ED = (\frac{\sqrt{3}}{2} + \frac{1}{2}) : \sin 75°$ and $BE : EF = (\sin 75° + \frac{\sqrt{3}}{2}) : \frac{1}{2}$.

5. In the previous problem, change $\overline{BF}$ from *angle bisector* to *median*. Show that $AE : ED = (3 + \sqrt{3}) : 3$ and $BE : EF = 2\sqrt{3} : 1$.

6. Prove that the altitudes of an acute triangle are concurrent using mass points. Review the clever method of showing this by forming a triangle for which the given triangle is the *medial* triangle and noticing that the perpendicular bisectors of the large triangle contain the altitudes of the medial triangle.

7. Let $\triangle ABC$ be a right triangle with a $30°$ angle at B and a $60°$ angle at A. Let $\overline{CD}$ be the altitude to the hypotenuse and let the angle bisector at B intersect $\overline{AC}$ at F and $\overline{CD}$ at E. Show that $BE : EF = (3 + 2\sqrt{3}) : 1$ and $CE : ED = 2 : \sqrt{3}$.

8. Let $\triangle ABC$ be a right triangle with $AB = 17$, $BC = 15$, and $CA = 8$. Let $\overline{CD}$ be the altitude to the hypotenuse and let the angle bisector at B intersect $\overline{AC}$ at F and $\overline{CD}$ at E. Show that $BE : EF = 15 : 2$ and $CE : ED = 17 : 15$.

9. Generalize the previous problems. Let $\triangle ABC$ be a right triangle with $AB = c$, $BC = a$, and $CA = b$. Let $\overline{CD}$ be the altitude to the hypotenuse and let the angle bisector at B intersect $\overline{AC}$ at F and $\overline{CD}$ at E. Show that $BE : EF = a : (a - c)$ and $CE : ED = c : a$.

10. Prove Ceva's Theorem for cevians that intersect in the interior of the triangle. *Three cevians of a triangle are concurrent if and only if the products of the lengths of the non-adjacent sides are equal.* (Hint: In $\triangle ABC$, let D, E, F be the intersection points of the cevians in sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. Let G be the intersection of the cevians, $AD = p$, $DB = q$, $BE = r$, and $EC = s$. Assign weight sq to A, sp to B, and rp to C).

11. Prove Menelaus' Theorem. *If a transversal is drawn across three sides of a triangle (extended if necessary), the product of the non-adjacent lengths are equal.*

4 More Problems

1. (**AHSME 1964 #35**) The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If $\overline{AD}$ is the altitude to the side of length 14, what is the ratio $HD : HA$?

2. (**AHSME 1965 #37**) Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find $\frac{EF}{FC} + \frac{AF}{FD}$.

3. (**AHSME 1975 #28**) In triangle ABC, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$ then find EG/GF.

4. (**AHSME 1980 #21**) In triangle ABC, $\angle CBA = 72°$, E is the midpoint of side AC and D is a point on side BC such that $2BD = DC$; $\overline{AD}$ and $\overline{BE}$ intersect at F. Find the ratio of the area of triangle BDF to the area of quadrilateral $FDCE$.

5. (**NYSML S75 #27**) In $\triangle ABC$, C' is on $\overline{AB}$ such that $AC' : C'B = 1 : 2$, and B' is on $\overline{AC}$ such that $AB' : B'C = 3 : 4$. If $\overline{BB'} \cap \overline{CC'} = P$ and if A' is the intersection of ray AP and $\overline{BC}$ then find $AP : PA'$.

6. (**NYSML F75 #12**) In $\triangle ABC$, D is on $\overline{AB}$ and E is on $\overline{BC}$. Let $\overline{CD} \cap \overline{AE} = K$ and let ray $BK \cap \overline{AC} = F$. If $AK : KE = 3 : 2$ and $BK : KF = 4 : 1$, then find $CK : KD$.

7. (**NYSML F76 #13**) In $\triangle ABC$, D is on $\overline{AB}$ such that $AD : DB = 3 : 2$ and E is on $\overline{BC}$ such that $BE : EC = 3 : 2$. If ray DE and ray AC intersect at F, then find $DE : EF$.

8. (**NYSML S77 #1**) In a triangle, segments are drawn from one vertex to the trisection points of the opposite side . A median drawn from a second vertex is divided , by these segments, in the continued ratio $x : y : z$. If $x \geq y \geq z$ then find $x : y : z$.

9. (**NYSML S77 #22**) A circle is inscribed in a 3-4-5 triangle. A segment is drawn from the smaller acute angle to the point of tangency on the opposite side. This segment is divided in the ratio $p : q$ by the segment drawn from the larger acute angle to the point of tangency on its opposite side. If $p > q$ then find $p : q$.

10. (**NYSML S78 #25**) In $\triangle ABC$, $\angle A = 45°$ and $\angle C = 30°$. If altitude $\overline{BH}$ intersects median $\overline{AM}$ at P, then $AP : PM = 1 : k$. Find k.

11. (**NYSML F80 #13**) In $\triangle ABC$, D is the midpoint $\overline{BC}$ and E is the midpoint of $\overline{AD}$. If ray BE and intersects $\overline{AC}$ at F. Find the value of $\frac{FE}{EB} + \frac{AF}{FC}$.

12. (**ARML 1989 T4**) In $\triangle ABC$, angle bisectors $\overline{AD}$ and $\overline{BE}$ intersect at P. If $a = 3$, $b = 5$, $c = 7$, $BP = x$, and $PE = y$, compute the ratio $x : y$, where x and y are relatively prime integers.

13. (**ARML 1992 I8**) In $\triangle ABC$, points D and E are on $\overline{AB}$ and $\overline{AC}$, respectively. The angle bisector of $\angle A$ intersects $\overline{DE}$ at F and $\overline{BC}$ at T. If $AD = 1, DB = 3, AE = 2$, and $EC = 4$, compute the ratio $AF : AT$

14. In the previous problem, if $AD = a, AB = b, AE = c$ and $AC = d$ then show that $\frac{AF}{AT} = \frac{ac(b+d)}{bd(a+c)}$.

15. (**AIME 1985 #6**) In triangle ABC, cevians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect at point P. The areas of triangles PAF, PFB, PBD and PCE are 40,30,35 and 84, respectively. Find the area of triangle ABC.

16. (**AIME 1988 #12**) Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$ and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$ then find abc.

17. (**AIME 1989 #15**) Point P is inside triangle ABC. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.

18. (**AIME 1992 #14**) In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find the value of $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

19. (Larson [**14**] problem 8.3.4) In triangle ABC, let D and E be the trisection points of BC with D between B and E. Let F be the midpoint of $\overline{AC}$, and let G be the midpoint of $\overline{AB}$. Let H be the intersection of $\overline{EG}$ and $\overline{DF}$. Find the ratio $EH : HG$.

20. Use nonconcurrency problems #4 and #5 to show that the triangle $\triangle JKL$ is one-seventh the area of $\triangle ABC$. Generalize the problem using points which divide the sides in a ratio of $1 : n$ to show the ratio of the areas is $(1-n)^3 : (1-n^3)$. This can be generalized even further using different ratios on each side. It is known as Routh's Theorem. See [**15**] [**16**] and [**17**].

5 Hints and Answers

1. $5 : 11$
2. $\frac{1}{2} + \frac{1}{1} = \frac{3}{2}$
3. $\frac{3}{2}$
4. $1 : 5$
5. $5 : 4$
6. $3 : 2$
7. $1 : 2$
8. $5 : 3 : 2$
9. $9 : 2$
10. $\frac{\sqrt{3}}{2}$
11. $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$
12. $2 : 1$
13. $5 : 18$
14. Split $cd(b-a)$ and $ab(d-c)$ at A and assign cad to B and cab to C.
15. 315
16. 441 (Show $\frac{d}{a+d} + \frac{d}{b+d} + \frac{d}{c+d} = 1$.)
17. 108 (Show CP:PF = 3:1. Draw a line segment from D to the midpoint of PB. Notice that it forms a 3-4-5 triangle which is one-eighth of the total area.
18. 94 (Assign weights of x, y, z to the vertices, find the ratios and multiply.)
19. $2 : 3$ (First draw $\overline{GC}$ intersecting $\overline{DF}$ at K. Find $CK : KG$. Now work on triangle DCG.)

6 References

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