

# Combinatorial Number Theory

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What is combinatorial number theory? Essentially, it's combinatorics, spiced up with some of the arithmetic properties of the integers. It has been characterized as the study of "structured sets of integers" – as opposed to algebraic, analytic, and other areas of number theory, which deal largely with algebraic relations and non-discrete properties of integers. If that makes no sense at the moment, the following sections should help to clarify.

Combinatorial number theory is, proportionately more than most other areas of mathematics, a recreational field - one studied lightly and without concern for applications. As a result, it is encountered substantially in the form of problems as well as in classical results. To try to reflect this, I've arranged a large number of problems here, with minimal introductions. Problems and results in combinatorial number theory really cannot be rigorously classified, but here is an effort at exposition of a few broad categories. These represent just some of the major areas of combinatorial number theory and are by no means intended to represent the field completely.

## 1 Divisibility issues

One basic area of study is related to the relation of divisibility between positive integers and its interplay with addition and ordering. Interpreting a number in terms of its prime factorization often simplifies questions of divisibility: a number is divisible by another if and only if each prime has an exponent in the first number at least as large as its exponent in the second; this formulation facilitates calculation of greatest common divisors and least common multiples. However, this interpretation of divisibility is far from trivializing many problems and results, and a variety of other techniques prove essential as well in the study of the seemingly simple notions of divisibility, relative primality, and so forth. Because the set of positive integers, together with the relation of divisibility, constitutes one of the basic examples of *partially ordered sets*, it is of considerable combinatorial importance.

1. Prove that if one chooses more than  $n$  numbers from the set  $\{1, 2, 3, \dots, 2n\}$ , then two of them are relatively prime.
2. Prove that if one chooses more than  $n$  numbers from the set  $\{1, 2, 3, \dots, 2n\}$ , then one number is a multiple of another. Can this be avoided with exactly  $n$  numbers? (Paul Erdős)
3. Does there exist an infinite sequence of positive integers, containing every positive integer exactly once, such that the sum of the first  $n$  terms is divisible by  $n$  for every  $n$ ?
4. (a) Suppose  $S$  is an infinite set of positive integers such that, for any finite nonempty subset  $T$  of  $S$ , there exists an integer  $> 1$  which divides every element of  $T$ . Show that there exists an integer  $> 1$  which divides every element of  $S$ .  
(b) Let  $k$  be a fixed positive integer. Show that there exists an infinite set  $S$  for which every subset  $T$  having at most  $k$  elements has a common divisor  $> 1$ , but no integer  $> 1$  divides every element of  $S$ .
5. Prove that, for each integer  $n \geq 2$ , there is a set  $S$  of  $n$  integers such that  $ab$  is divisible by  $(a - b)^2$  for all distinct  $a, b \in S$ . (USA, 1998)
6. Given 81 positive integers all of whose prime factors are in the set  $\{2, 3, 5\}$ , prove that there are 4 numbers whose product is the fourth power of an integer. (Greece, 1996)
7. Show that there exists a set  $A$  of positive integers with the following property: For any infinite set  $S$  of primes there exists an integer  $k \geq 2$  and two positive integers  $m \in A$  and  $n \notin A$  such that each of  $m, n$  is a product of  $k$  distinct elements of  $S$ . (IMO, 1994)

8. Let  $S$  be a finite set of integers, each greater than 1. Suppose that, for each integer  $n$ , there is some  $s \in S$  such that the greatest common divisor of  $s$  and  $n$  equals either 1 or  $s$ . Show that there exist  $s, t \in S$  whose greatest common divisor is prime. (Putnam, 1999)
9. Let  $S = \{1, 2, 3, \dots, 280\}$ . Find the smallest positive integer  $n$  such that every  $n$ -element subset of  $S$  contains five numbers which are pairwise relatively prime. (IMO, 1991)
10. If the set  $A$  consists of  $n$  positive integers, show that the set  $\{ab/\gcd(a, b)^2 : a, b \in A\}$  contains at least  $n$  members. (*Amer. Math. Monthly*, 1999)

## 2 Partitions of sets of integers

Another important area of combinatorial number theory is the study of what happens when the positive integers, or some finite subset thereof, are distributed among a bunch of smaller sets, often phrased in terms of “coloring” the integers. It is of particular interest to define a “largeness” or “density” property for sets of positive integers, and to show that, when the positive integers are divided into finitely many subsets, some subset is “large.” For example, a result of Schur states that, for every positive integer  $k$ , there is some  $n$  with the following property: when the integers from 1 through  $n$  are divided into  $k$  subsets, some subset contains three distinct numbers such that one is the sum of the other two. Another significant example is Van der Waerden’s theorem that, if the set of all positive integers is divided into finitely many subsets, there is some subset which contains arbitrarily long arithmetic progressions. In addition to finding properties that one subset must have, one also studies relations between the subsets. This study is particularly useful when one restricts one’s attention to partitions of the integers into particular types of subsets, such as arithmetic progressions. A variety of combinatorial and number theoretic techniques - construction, induction, contradiction, direct proofs - are used in this area.

1. Is it possible for the numbers  $1, 2, \dots, 100$  to be the terms of 12 geometric progressions? (Russia, 1995)
2. You want to color the integers from 1 to 100 so that no number is divisible by a different number of the same color. What is the smallest possible number of colors you must have?
3. The set of positive integers is partitioned into finitely many subsets. Show that some subset  $S$  has the following property: for every positive integer  $n$ ,  $S$  contains infinitely many multiples of  $n$ . (BMC contest, 1999)
4. A set of  $S$  positive integers is called a *finite basis* if there exists some  $n$  such that every sufficiently large positive integer can be written as a sum of at most  $n$  elements of  $S$ . If the set of positive integers is divided into finitely many subsets, must one of them necessarily be a finite basis?
5. Prove that the set of positive integers can be divided into an infinite number of infinite sets so that the following holds: if  $x, y, z, w$  all belong to the same subset, then  $x - y$  and  $z - w$  belong to the same subset if and only if  $x/y = z/w$ . (Colombia, 1997)
6. Suppose that the positive integers have been colored in four colors - red, green, blue, and yellow. Let  $x$  and  $y$  be odd integers of different absolute values. Show that there exist two numbers of the same color whose difference has one of these values:  $x, y, x - y$ , or  $x + y$ . (IMO Proposal, 1999)
7. The set of all integers is partitioned into (disjoint) arithmetic progressions. Prove that some two of them have the same common difference. (Sasha Schwartz)
8. The set of positive integers is divided into finitely many (disjoint) subsets. Prove that one of them, say  $A_i$ , has the following property: There exists a positive number  $m$  such that, for every  $k$ , one can find numbers  $a_1, a_2, \dots, a_k$  in  $A_i$  with  $0 < a_{j+1} - a_j \leq m$  for all  $j, 1 \leq j \leq k - 1$ . (IMO Proposal, 1990)
9. Determine whether there exists an integer  $n > 1$  satisfying the following: The set of positive integers can be partitioned into  $n$  nonempty subsets so that an arbitrary sum of  $n - 1$  elements, one taken from each of any  $n - 1$  of the subsets, lies in the remaining subset. (IMO Proposal, 1995)

### 3 Additive problems

One of the largest areas of combinatorial number theory - and one of the broadest, as it connects not only with combinatorics but also analysis and algebra - is additive number theory: the study of what happens when sets of integers are added together. One of the most famous unsolved problems in number theory is an additive one: the Goldbach conjecture, which says that every even number greater than 4 is the sum of two odd primes. Many combinatorial and algebraic techniques prove extremely valuable here; most notable is the pigeonhole principle, but other methods of approaching problems include classifying sets of numbers to be added, the use of minimal and maximal elements, counting sets in multiple ways, and the representation of a sum of numbers as a difference of two other sums. These problems furnish a few examples.

1. Prove that any set of  $n$  integers has a nonempty subset whose sum is divisible by  $n$ .
2. Let  $A$  be a subset of  $\{0, 1, 2, \dots, 1997\}$  containing more than 1000 elements. Prove that  $A$  contains either a power of 2, or two distinct elements whose sum is a power of 2. (Ireland, 1997)
3. Fifty numbers are chosen from the set  $\{1, \dots, 99\}$ , no two of which sum to 99 or 100. Prove that the chosen numbers must be  $50, 51, 52, \dots, 99$ . (St. Petersburg, 1997)
4. For any set  $A$  of positive integers, let  $n_A$  denote the number of triples  $(x, y, z)$  of elements of  $A$  such that  $x < y$  and  $x + y = z$ . Find the maximum value of  $n_A$ , given that  $A$  contains seven elements. (Norway, 1997)
5. Given is a list of  $n$  positive integers whose sum is less than  $2n$ . Prove that, for any positive integer  $m$  not exceeding the sum of these  $n$  integers, one can choose some of the integers so that their sum is  $m$ .
6. Let  $n$  be a positive integer, and let  $X$  be a set of  $n + 2$  integers, each of absolute value at most  $n$ . Show that there exist three distinct numbers  $a, b, c$  in  $X$  such that  $a + b = c$ . (India, 1998)
7. Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint nonempty subsets whose members have the same sum. (IMO, 1972)
8. Find the greatest positive integer  $n$  for which there exist  $n$  nonnegative integers  $x_1, x_2, \dots, x_n$ , such that for any sequence  $\epsilon_1, \epsilon_2, \dots, \epsilon_n$  of elements of  $\{-1, 0, 1\}$ , not all zero,  $n^3$  does not divide  $\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n$ . (Romania, 1996)
9. Let  $n \geq 2$ . Show that there exists a subset  $S$  of  $\{1, 2, \dots, n\}$  having at most  $2\lfloor\sqrt{n}\rfloor + 1$  elements such that every positive integer less than  $n$  is representable as a difference of two elements of  $S$ . (Romania, 1998)
10. Show that any positive integer can be expressed as a sum of terms of the form  $2^a 3^b$  such that none of these terms is divisible by any other. (Paul Erdős)
11. Let  $x_1, x_2, \dots, x_{19}$  be positive integers less than or equal to 93. Let  $y_1, y_2, \dots, y_{93}$  be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some  $x_i$  equal to a sum of some  $y_j$ . (Putnam, 1993)
12. Let  $p$  be an odd prime. Determine the number of  $p$ -element subsets of  $\{1, 2, \dots, 2p\}$  such that the sum of the elements is divisible by  $p$ . (IMO, 1995)
13. Given  $2n - 1$  integers, prove that one can choose exactly  $n$  of them whose sum is divisible by  $n$ . (Paul Erdős)

### 4 Partitions and related topics

The deepest and most serious area of research in combinatorial number theory is concerned with partitions of a positive integer. A *partition* is a representation of an integer as a sum of other positive integers, called the *parts*. Two partitions with the same parts in a different order are considered the same. For example, 4

has 5 partitions:  $4, 3 + 1, 2 + 2, 2 + 1 + 1$ , and  $1 + 1 + 1 + 1$ . Many colorful results are concerned with the number of partitions of a positive integer or the number of partitions satisfying a particular condition. A famous result of Ramanujan (who did extensive work on partitions) asserts that the number of partitions of a number of the form  $5k + 4$  is always divisible by 5. Also important is Euler's "pentagonal number theorem," which provides a recurrence relation for calculating the number  $p(n)$  of partitions of  $n$ : it equals the sum of  $(-1)^{m+1}p(n - \frac{m(3m-1)}{2}) + p(n - \frac{m(3m+1)}{2})$  over all positive integers  $m$ , where we take  $p(0) = 1$  and  $p(n) = 0$  if  $n < 0$ . Partitions played a crucial role in the recent proof of the alternating sign matrix conjecture.

Unlike many other areas of combinatorial number theory, there are a few specific techniques which are of considerable importance in examining partitions. One is the use of Ferrers diagrams, a method of representing partitions graphically; also useful is the technique of bijection, where one finds a function that turns one type of partition into another type of partition and thereby shows that the number of partitions of the first type equals the number of partitions of the second type. By far the most significant tool, however, is the use of the *generating function*. A generating function is a power series (or "infinite polynomial") in which the coefficients represent a significant sequence of numbers. By doing algebraic operations on entire series, one can obtain information about the individual coefficients. Sequences where the  $n$ th coefficients represents information about partitions of  $n$  happen to be particularly amenable to this kind of manipulation. There are non-partition-related problems as well which can be solved with generating-function techniques, and a few such problems occur below.

1. Let  $m$  and  $n$  be positive integers with  $n > \frac{1}{2}m(m+1)$ . Show that the number of partitions of  $n$  into  $m$  distinct parts equals the number of partitions of  $n - \frac{1}{2}m(m+1)$  into at most  $m$  parts. (Korea, 1995)
2. Show that the number of partitions of a positive integer  $n$  into distinct parts equals the number of partitions of  $n$  into odd parts.
3. Find, in terms of  $m$  and  $n$ , the number of partitions of (arbitrary) positive integers into at most  $m$  parts, each less than or equal to  $n$ .
4. Consider partitions of a positive integer  $n$  into (not necessarily distinct) powers of 2. Let  $f(n)$  be the number of such partitions with an even number of parts, and let  $g(n)$  be the number of such partitions with an odd number of parts. Find all  $n$  for which  $f(n) = g(n)$ .
5. Let  $p(n)$  denote the number of partitions of the integer  $n$ , and let  $f(n)$  denote the number of partitions of positive integers into distinct parts in which, when the parts are listed in descending order,  $n$  is the sum of the 1st, 3rd, 5th, etc. parts. For example,  $p(5)$  counts the 7 partitions  $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$ , and  $f(5)$  counts the 7 partitions  $5, 5+1, 5+2, 5+3, 5+4, 4+3+1, 4+2+1$ . Prove that  $p(n) = f(n)$  for every positive integer  $n$ . (*Amer. Math. Monthly*, 1997)
6. For any set  $A$  of positive integers, we can form a list of all possible sums of two distinct members of  $A$ . (Writing a list, rather than a set, means that a number may occur multiple times, according to how many ways it can be represented as such a sum.) Suppose that two distinct sets  $A, B$  produce the same list. Show that the number of elements in each set is a power of 2. (Paul Erdős and John Selfridge)
7. The numbers  $1, 2, \dots, 2n$  are divided into 2 groups of  $n$  numbers. We form a list of the remainders formed by dividing the sums  $a + b$  by  $2n$ , where  $a, b$  are in the same group (and may be equal). Prove that the  $n^2$  remainders from one group are equal, in some order, to the  $n^2$  remainders of the other group. (St. Petersburg, 1996)
8. For any partition  $\pi$  of a positive integer  $n$ , define  $A(\pi)$  to be the number of 1's which appear in  $\pi$ , and define  $B(\pi)$  to be the number of distinct integers which appear in  $\pi$ . (For example, if  $n = 13$  and  $\pi$  is the partition  $5 + 2 + 2 + 2 + 1 + 1$ , then  $A(\pi) = 2, B(\pi) = 3$ .) Prove that, for fixed  $n$ , the sum of  $A(\pi)$  over all partitions  $\pi$  of  $n$  equals the sum of  $B(\pi)$  over all partitions  $\pi$  of  $n$ . (USA, 1986)
9. For each positive integer  $n$ , let  $f(n)$  denote the number of partitions of  $n$  into powers of 2. Prove that, for any  $n \geq 3$ ,  $2^{n^2/4} < f(2^n) < 2^{n^2/2}$ . (IMO, 1997)