BERKELEY MATH CIRCLE MONTHLY CONTEST #5, DUE FEBRUARY 20, 2000

1. Given are $n+1$ real linear equations in n variables (of the form $a_1x_1+a_2x_2+\cdots+a_nx_n = a$). Prove that each $=$ sign can be replaced with either $\leq$ or $\geq$ so that the resulting $n+1$ inequalities have the following property: for every choice of real numbers $x_1, x_2, \ldots, x_n$, at least one inequality is true.
2. Let m, n be positive integers. Suppose that a given rectangle can be tiled (without overlaps) by a combination of horizontal $1 \times m$ strips and vertical $n \times 1$ strips. Show that it can be tiled using just one of the two types.
3. The set S is a finite subset of $[0, 1]$ with the following property: for all $x \in S$, there exist $a, b \in S \cup \{0, 1\}$ with $a, b \neq x$ such that $x = (a + b)/2$. Prove that all the numbers in S are rational.
4. Certain cities are connected by roads connecting pairs of them. The roads intersect only at the cities. A subset of the roads is called *important* if destroying those roads would make it so that there are two cities such that it is impossible to go from the first to the second. A subset S of the roads is called *strategic* if it is important and if no proper subset of S is important. Let S and T be distinct strategic sets of roads. Let U be the set of roads that are in either S or T, but not both. Prove that U is important.
5. Let $ABCDEF$ be inscribed in circle k so that $AB = CD = EF$. Assume that diagonals AD, BE, CF intersect in point Q, and let $P = AD \cap CE$. Prove $CP/PE = AC^2/CE^2$.

Please write solutions to different problems on separate pages. At the top of each page, write your name, school, city, contest number, and problem number. Remember that these problems are not to be discussed with anyone until after the due date. Please go to
`http://www.math.berkeley.edu/~stankova/MathCircle/Joyce/index2.html`
for more information about the contest.

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