Berkeley Math Circle

Take-Home Contest #2 – Solutions

1. A 6×6 square is covered by nonoverlapping dominos (2×1 rectangles, placed horizontally or vertically). Prove that there must be a horizontal line or a vertical line that passes through the interior of the big square, but which does not cut the interior of any domino.

   Solution: Consider all the grid lines of the big square. If some such line is intersected by d dominos, we claim d is even. Proof: Looking at the portion of the board on one side of the line, we find that the number of squares there is divisible by 6 and so is even; on the other hand, each of the d dominos covers exactly 1 square there and any other domino covers either 0 or 2, so the number of squares has the same parity as d; hence, d is even.

   Now if each of the 5 horizontal and 5 vertical lines were intersected by some domino, the claim implies that each would intersect at least 2 dominos. Thus we would have 20 intersections, and since a domino cannot cross more than one line, this gives 20 distinct dominos. This is a contradiction since we only use 18 dominos to cover the square.

2. The set of positive integers is parititioned into finitely many subsets. Show that some subset S has the following property: for every positive integer n, S contains infinitely many multiples of n.

   Solution: Let the subsets be $S_1, S_2, \ldots, S_k$. Suppose the statement is false and seek a contradiction. Then for each S_i there exists some n_i such that S_i contains only finitely many multiples of n_i. Let $n = n_1 n_2 \cdots n_k$; then every multiple of n is a multiple of each n_i and so each S_i can contain only finitely many multiples of n. But this means that the k sets together contain only finitely many multiples of n, and since they partition the positive integers (which contain infinitely many multiples of n), we have our contradiction.

3. The Cannibal Club of California (CCC) had 30 members yesterday morning – but that was before their festive annual dinner! After the dinner, it turned out that among any six members of the club, there was a pair one of whom ate the other. Prove that at least six members of the CCC are now nested inside one another.

   Solution: We must assume that nobody was eaten by more than one person for the problem statement to make sense. To each cannibal we assign a numerical "depth" as follows: the depth of cannibal C is the largest integer n such that there exist cannibals $C_1, C_2, \ldots, C_n = C$ such that C_i ate C_{i+1} for $i = 1, 2, \ldots, n-1$. (A cannibal who was eaten by noone has depth 1. Also note that depth is definable: there is an upper bound on the value of n, since any chain of length greater than 30 would contain some cannibal twice, an impossibility; hence there is a maximum value of n for which chains exist.) Note that no cannibal ate another of the same depth, since the inner cannibal always has higher depth. Now, if we can find a cannibal of depth ≥ 6 we are done, so assume that the only depths which occur are $1, 2, 3, 4, 5$. By the pigeonhole principle, some depth was assumed by at least 6 cannibals; from the given, one of these six ate another. But we know this is impossible, so our assumption was wrong and some higher depth does occur.

   Remark: Several students were puzzled about the wording of this problem. We see that, in actuality, not much information is needed (e.g. whether "eating" is defined transitively) to find a general solution.

4. Let E be an ellipse that is not a circle. For which $n \geq 3$ is it possible to inscribe a regular n-gon in E? (For each n, either show how to construct such an n-gon or prove that none exists.)

   Solution: We claim $n = 3, 4$ are the only solutions. To see that 3 is possible, let P be the endpoint of one of the axes, and draw two lines at angles of $\pi/6$ to that axis through P. By symmetry, these lines intersect the ellipse at points Q,R equidistant from P, so PQR is an equilateral triangle. To see that 4 is possible, draw the four lines through the center of E at angles of $\pi/4$ to its axes; by symmetry, they intersect the ellipse at points which are equidistant from (and subtend equal angles at) its center and therefore form a square.

Now, the vertices of any regular polygon lie on a common circle. By Bezout's theorem, this circle can only intersect E in 2^2 points (since both are curves of degree 2), so only 4 vertices of our polygon can lie on the ellipse, and we cannot have $n > 4$. (Alternatively, one can see this using the fact that 5 distinct points determine a unique conic section - that conic cannot be both E and a circle.)

5. Prove that

$$\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) = \sqrt{11}.$$

Solution: We work with complex roots of unity. Let $\zeta = \cos\pi/11 + i\sin\pi/11$; then we know that $\zeta^n = \cos n\pi/11 + i\sin n\pi/11$ and, in particular, $\zeta^{11} = -1$. Now, the given expression is clearly positive (since each term is positive), so we need only check that its square is 11. To do that, observe that $\sin 2\pi/11 = (\zeta^2 + \zeta^9)/2i$, $\sin 3\pi/11 = (\zeta^3 + \zeta^8)/2i$, and $\cos 3\pi/11 = (\zeta^3 - \zeta^8)/2$; hence our objective is to show that

$$\left(2\frac{\zeta^2 + \zeta^9}{i} + \frac{\zeta^3 + \zeta^8}{(\zeta^3 - \zeta^8)i}\right)^2 = 11$$

or, equivalently, that $[2(\zeta^2 + \zeta^9)(\zeta^3 - \zeta^8) + (\zeta^3 + \zeta^8)]^2 = -11(\zeta^3 - \zeta^8)^2$. The expression inside brackets on the left multiplies out to $2\zeta^5 + 2\zeta^{12} - 2\zeta^{10} - 2\zeta^{17} + \zeta^3 + \zeta^8 = -2\zeta^{10} + \zeta^8 + 2\zeta^6 + 2\zeta^5 + \zeta^3 - 2\zeta$; when we square this and collect terms (remembering again that $\zeta^{11} = -1$), we obtain

$$4\zeta^{10} - 4\zeta^9 + 4\zeta^8 - 4\zeta^7 - 7\zeta^6 + 7\zeta^5 + 4\zeta^4 - 4\zeta^3 + 4\zeta^2 - 4\zeta - 18$$

$$= 4(\zeta^{10} - \zeta^9 + \zeta^8 - \cdots + 1) + 11(\zeta^5 - 2 - \zeta^6).$$

Since the first expression in parentheses is $(\zeta^{11} + 1)/(\zeta + 1) = 0$ and the second is $-(\zeta^3 - \zeta^8)^2$, we have what we wanted.

Remark: As one student observed, a similar proof shows that $\tan(3k\pi/11) + 4\sin(2k\pi/11) = \pm\sqrt{11}$ for any integer k not divisible by 11. A related result of Gauss states that, for any odd prime p, if $\zeta = \cos 2\pi/p + i\sin 2\pi/p$, then

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\zeta^j = \begin{cases} \sqrt{p} & \text{if } p = 4k+1 \\ i\sqrt{p} & \text{if } p = 4k+3, \end{cases}$$

where we define $(j/p) = 1$ if there exists an integer n with $n^2 - j$ divisible by p and -1 otherwise.