BERKELEY MATH CIRCLE MONTHLY CONTEST #1, DUE 10/10/99

1. Let k and n be positive integers such that $k < 2^{n+1} - 1$. Prove that there is a sum of *exactly* n powers of 2 that is divisible by k. (Example: if $k = 9$ and $n = 4$, then $2 + 4 + 16 + 32$ is divisible by 9.)
2. Ten tourists visiting a tropical island are captured by one of the natives, who happens to be a cannibal. The cannibal explains to them that it is the custom on his island to give prisoners a test before eating them. Tomorrow he will line them up in single file, and he will place a black or white hat on each of them, so that each tourist can see only the colors of the hats of the tourists ahead in the line. Then, starting from the rear of the line, the cannibal will ask each tourist to guess the color of his hat, based on the colors of the hats ahead, and based on any earlier guesses that were made by tourists behind him. (The tourists are not allowed to signal each other in any other way.) If a tourist names the wrong color, the cannibal silently eats that tourist before moving on to the next. At the end, any uneaten tourists are freed. Show that if the tourists are allowed to plan a strategy the night before they are tested, they can guarantee that at least nine of them will escape.
3. Let n be an integer greater than 2. Positive real numbers x and y satisfy $x^n = x + 1$ and $y^{n+1} = y^3 + 1$. Prove that $x < y$.
4. Let $z = \cos 2\pi/n + i \sin 2\pi/n$ where n is a positive odd integer. Prove that
$$\frac{1}{1+z} + \frac{1}{1+z^2} + \frac{1}{1+z^3} + \cdots + \frac{1}{1+z^n} = \frac{n}{2}.$$
5. An apple is in the shape of a ball of radius 31mm. A worm gets into the apple and digs a tunnel of total length 61mm, and then leaves the apple. (The tunnel need not be a straight line.) Prove that one can cut the apple with a straight slice through the center so that one of the two halves is not rotten.

Please write solutions to different problems on separate pages. At the top of each page, write your name, school, city, contest number, and problem number. Please go to `http://www.math.berkeley.edu/~stankova/MathCircle/Joyce/index2.html` for more information about the contest.

Date: September 12, 1999.