-bb-error = =

ROOKIES–VETERANS CONTEST
December 20 1998

VETERANS SECTION

From the "Trigonometry Kingdom"

Problem 1. Let α and β be two angles. As every baby knows,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Prove this identity (for the benefit of us, the grown-ups :-)).

(Hint: From the zillions of ways in which this identity can be proved, you may consider applying Ptolemy's theorem to some quadrilateral one of whose diagonals is a diameter of your circle.)

Problem 2. Let α be an angle. Find a closed formula for the sum

$$C_n = \cos\alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos n\alpha.$$

For example, a closed formula for the sum $1+2+3+\cdots+n$ is $n(n+1)/2$.

(Hint: Here you can show off your knowledge of complex numbers, De Moivre's formula and geometric series - a big arsenal of tools!!)

From the "Don't Try This at Home" Department

Problem 3. Pasha has an egg from an exotic animal, and through research Pasha finds out that he needs to boil the egg for exactly 45 minutes. As we know, the world is not perfect, and it turns out that Pasha doesn't have a watch. What he has is: two wires (ordinarily used for explosives), each of which can burn for exactly 60 minutes. The wires do not have the same consistency everywhere and correspondinly some parts of them may burn faster than others. Can Pasha measure exactly 45 minutes using these two wires and cook his egg appropriately?

(O.K. I grant you that Pasha has matches to light up the wires.)

ROOKIES SECTION

What We Always Wanted to Know about Rivers and Bridges, but were Afraid to Ask

Problem 1. There are two farms situated on two different sides of a very wide river. Find where the farmers should build a bridge across the river so that the total distance from one farm to the other (across the river through the bridge) is minimal.

(Hint: Everyone agrees that the problem would be trivial if the river were of zero width. Can you reduce the original problem to this trivial one?)

-bb-error = =

FIG. 1-3

Problem 2. There is a wide river and a narrow old bridge across it. In order to cross the bridge during the night, one has to carry a flashlight, and at most two people can cross the bridge together (using the same flashlight.) The bridge is so unsafe that noone can even stay on it without a flashlight. If two people cross together, they move at the speed of the slower person. One night during a war, four soldiers end up on one side of the bridge while running from the other army. The soldiers need to cross the bridge in at most 17 minutes in order to escape. However, they have only one flashlight. Due to some injuries it is going to take each soldier 1, 2, 5 and 10 minutes, respectively, to cross the bridge individually. It is easy to see that the soldiers can cross the bridge in 19 minutes. Would they be able to cross it in 17 minutes and thus escape?

(Hint: No hint here, dudes! You are on your own! However, be aware that no funny business is allowed here, e.g. no swimming (we might as well assume that there are crocodiles in the river), none of the soldiers is an olympic champion in flashlight throwing (we might as well assume that the river is really, really, really wide), and finally, the flashlight cannot swim, OK, cannot float!)

From the "Agricultural Department"

Problem 3. A farmer has a goat and a semidisc of grass, surrounded by a poisonous plant. How can the farmer, using finitely many poles and finitely many ropes (of whatever length he desires), tie up the goat in such a way that the goat can eat everywhere inside the semidisc, but nowhere outside it?

(Warning: The poles are really short: so, for instance, you can't make a fence from them, for the goat will be able to jump over it. Also, don't kill the goat and don't do anything nasty to it - please, be nice and reasonable mathematicians! :-))