BERKELEY MATH CIRCLE, January 17, 1999

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"PROOFS" Problem Set #3[1]

Prove the following statements using the Pigeon-hole Principle:

1. Among a group of 367 people, at least 2 persons will have the same birthday.
2. Among the students in this Math Circle, at least 2 of them will have their birthdays fall on the same day of the month.
3. Suppose 51 numbers are chosen from $1, 2, 3, \ldots, 99, 100$. Show that there are two which do not have a common divisor.
4. Suppose 51 numbers are chosen from $1, 2, 3, \ldots, 99, 100$. Show that there are two such that one divides the other.
5. Show that among any 9 distinct real numbers, there are two, say a and b, such that $0 < \frac{a-b}{1+ab} < \sqrt{2} - 1$.
6. Suppose a triangle can be placed inside a unit square in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than 1.
7. A region M in the plane has area greater than 1. Prove that there are two points (a, b) and (c, d) in M such that $a - c$ and $b - d$ are both integers.
8. In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the kth tallest boy and the kth tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.
9. Prove that from any sequence of 1999 real numbers, one can choose a block of consecutive terms whose sum differs from an integer by at most .001.
10. Eight students took part in a contest with eight problems. Each problem was solved by 5 students:
    (a) Prove that there were two students who between them solved all eight problems.
    (b) Prove that this is not necessarily the case if 5 is replaced by 4.
11. Prove that among any six integers there will be a pair whose sum or difference is divisible by 9.
12. Suppose 9 points with integer coordinates in three dimensional space are chosen. Show that one of the segments with endpoints selected from the 9 points must contain a third point with integer coordinates.
13. Eleven numbers are chosen from $1, 2, 3, \ldots, 99, 100$. Show that there are two nonempty disjoint subsets of these eleven numbers whose elements have the same sum.
14. Show that among any six people, either there are three who know each other or there are three, no pair of which knows each other.

Use the Infinite Descent Method to prove or solve the following:

15. Find all pairs of positive integers x and y satisfying the equation $x^2 - 2y^2 = 0$.
16. Show that the equation $x^2 + y^2 + z^2 = 2xyz$ has no integral solutions except $x = y = z = 0$.
17. Do there exist positive integers x and y such that $x + y$, $2x + y$ and $x + 2y$ are perfect squares?
18. There are no integer solutions of $x^4 + y^4 = z^2$.
19. There are no integer solutions of $x^4 - y^4 = z^2$.
20. Prove that there are no positive integers m and n such that $2(m^2 + mn + n^2)$ is a perfect square.
21. The set $Z \times Z$ is called the *plane lattice*. Prove that for $n \neq 4$ there exist no regular n-gon with lattice points as vertices.
22. Prove that the following equation has no solutions in positive integers: $x^2 + y^2 + z^2 + u^2 = 2xyzu$.