BERKELEY MATH CIRCLE, September 27, 1998

How to Write Proofs Part I[1]

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Problems for discussion.

1. Prove that $\frac{n^2}{2} < 1 + 2 + 3 + 4 + \cdots + n < \frac{(n+1)^2}{2}$.
2. Given a set of two million points on the plane. Prove that there exists a straight line that separates this set into two subsets of exactly one million points each.
3. (Fermat's Little Theorem) If a natural number a is not divisible by a prime number p, that is $p \not| a$, then $a^{p-1} \equiv 1 \pmod{p}$.
4. There exist arbitrarily large gaps between consecutive prime numbers.
5. (Chinese Remainder Theorem)
   Consider the system of k linear congruences: $x \equiv b_i \pmod{m_i}$ $(i = 1, 2, 3, ..., k)$, where $(m_i, m_k) = 1$ for all $i \neq j$. Let $m = m_1 m_2 m_3 \cdots m_k$. This system has a unique solution $x_0 \pmod{m}$.
6. Prove there are infinitely many prime numbers.
7. A Fermat prime is a prime of the form $2^{2^n} + 1$. How many Fermat primes are there?
8. Find a formula $P(n)$ so that it produces distinct primes for each natural number n.
9. (Goldbach Conjecture) Is every even number greater than two a sum of two primes?
10. Twin primes are primes that differ by two. Are there infinitely many twin primes?
11. A regular p-gon, where p is prime, is constructible if and only if p is a Fermat prime.
12. Prove that $\sqrt{2}$ is not a rational number.

"PROOFS" Problem Set #1

1. There are infinitely many primes of the form $6n + 5$.
2. If $(a, m) = 1$, then $ax \equiv b \pmod{m}$ has a unique solution.
3. If $2^n - 1$ is prime then n is prime.

4. Given any three odd positive integers, it is possible to find a fourth odd positive integer such that the sum of the squares of all four numbers is a perfect square.

5. The sequence $2^n - 3, n = 2, 3, 4, ...$ contains infinitely many terms divisible by 5 and infinitely many terms divisible by 13, but no terms divisible by (5)(13).

6. Is it possible for a sum of reciprocals of distinct primes to add up to an integer?

7. Is it possible for a sum of reciprocals of the first $n,\ n > 1$, natural numbers to add up to an integer?

8. Factor $a^4 + 4b^4$.

9. For natural numbers n, when is $n^4 + 4^n$ prime?

10. Is $4^{545} + 545^4$ prime?

11. Prove or disprove that there are two rational numbers such that $p + q = pq = 1$.

12. Prove or disprove that there are three rational numbers such that $p+q+r = pqr = 1$.

13. Prove or disprove that there are four rational numbers such that $p + q + r + s = pqrs = 1$.

14. If the sum of two positive integers is 2310, then their product is not divisible by 2310.

15. Let a, b, c and p be real numbers, with a, b, c not all equal, such that $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = p$. Determine all possible values of p and prove that $abc + p = 0$.

16. If all of the positive integers are separated into two subsets with no common elements, then one of these two subsets must contain a three term arithmetic progression.

17. Find all nonnegative integers x, y satisfying $(xy - 7)^2 = x^2 + y^2$.

18. Find all prime numbers p such that $2^p + p^2$ is also a prime.

19. Prove or disprove that there is a positive integer n such that $\sqrt{n-1} + \sqrt{n+1}$ is a rational number.

20. Find the smallest integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

21. $4^n + 2$ is divisible by 6 for every positive integer n.

22. If a, b, c are positive numbers such that $abc = 1$, then

$$\frac{a}{ab + a + 1} + \frac{b}{bc + b + 1} + \frac{c}{ca + c + 1} = 1.$$

23. If in a set of 21 numbers, the sum of any 10 is less than the sum of the other 11, then all of the numbers are positive.

24. The closed curve $ABCDEFGHA$ consists of 8 straight line segments $AB, BC, CD, DE, EF, FG, GH$ and HA. The 8 points A, B, C, D, E, F, G and H lie at the vertices of a cube. The curve does not intersect itself. Then at least one of the segments of the curve coincides with an edge of the cube.

25. If quadrilateral $ABCD$ is inscribed in a circle with center O and $\overline{AC} \perp \overline{BD}$, then the area of $AOCB =$ the area of $AOCD$.

26. Triangle ABC is equilateral and P is a point in the interior of the triangle such that the perpendicular distances to the sides, $PQ = 6$, $PR = 8$, and $PS = 10$. Find the area of the triangle.