# PROBLEMS OF MARRIAGE

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## 1. The best strategy to find the best spouse.

A person $A$ is looking for a spouse, so $A$ starts dating. After $A$ dates the person $B$, $A$ decides whether s/he wants to marry $B$ or to reject $B$ and start dating somebody else. Of course, after having dated two different persons $B$ and $C$, $A$ can tell which one was a better choice (but not before).

Assume that during life time $A$ will meet $n$ people who potentially can become $A$'s spouse. $A$ plays it honest – $A$ can get married only once and $A$ cannot get married to a person who was already rejected by $A$. In the end $A$ does not think s/he is a looser only if $A$ got married to the best person out of all $n$ people s/he has met.

**Question 1.** What is the best strategy for $A$ to get the best partner?

**Question 2.** What are $A$'s chances of getting the best partner?

**Example.** The person $A$ has a simple strategy – s/he marries the first date. Then the chances of $A$ to win are $1/n$.

Let us call $St_k$ the following strategy. $A$ meets first $k$ people and rejects them all. Then $A$ marries the first person who is better than all the previous ones. Of course, here $k$ can be $0, 1, 2, \ldots, n-1$.

**Question 1'.** Which of the strategies $St_k$ is the best?

**Question 2'.** What are the chances of $A$ of getting the best partner if s/he follows strategy $St_k$?

**Theorem 1.** *If $A$ uses strategy $St_k$ then his/her chances are*

$$C(k) = \frac{1}{n}\left(1 + \frac{k}{k+1} + \cdots + \frac{k}{n-1}\right).$$

*Proof*: Let us fix $k, n$ and compute the chances of $A$ to win.

a) There is a $1/n$ chance that the best partner will be number $k+1$ – in this case $A$ wins.

b) There is a $1/n$ chance that the best partner is number $k+2$. In this case, $A$ wins if s/he rejects the date number $k+1$. It means that the best partner out of the first $k+1$ partners was not number $k+1$. Chances for that are $k/(k+1)$. Total chances in this case are $(1/n) * (k/(k+1))$.

c) There is a $1/n$ chance that the best partner is number $k+3$. In this case, $A$ wins if s/he rejects the dates number $k+1$ and $k+2$. It means that the best partner out of the first $k+2$ partners was neither number $k+1$ nor $k+2$. Chances for that are $k/(k+2)$. Total chances in this case are $(1/n) * (k/(k+2))$.

Continuing this argument we prove the Theorem. $\square$

**Example.**

a) $n = 2$: $C_0 = C_1 = 1/2$.
b) $n = 3$: $C_0 = 1/3$, $C_1 = 1/2$, $C_2 = 1/3$.
c) $n = 4$: $C_0 = 1/4$, $C_1 = 11/24$, $C_2 = 10/24$, $C_3 = 1/4$.
d) $n = 5$: $C_0 = 1/5$, $C_1 = 25/60$, $C_2 = 26/60$, $C_3 = 21/60$, $C_4 = 1/5$.

Now, we want to estimate, given $n$, for which $k$ the number $C_k$ is the greatest. To do that consider the difference

$$D_k := C_k - C_{k-1} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1} - 1\right).$$

Clearly, $D_k$ are positive for $k = 1, \ldots, k_0$ and negative for $k = k_0 + 1, k_0 + 2, \ldots, n-1$. So, $C_k$ is the greatest when $k = k_0$.

For $k_0$ we have (approximately)

$$\frac{1}{k_0} + \frac{1}{k_0+1} + \cdots + \frac{1}{n-1} \approx 1. \tag{1}$$

**Lemma 1.**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = \ln n + \gamma + a(n),$$

*where* $0 < \gamma < 1$ *is some constant and* $a(n)$ *is a small number for large* $n$*.*

*Proof*: a) Consider the graph of function $y = 1/x$. Then $1/2 + 1/3 + \cdots + 1/(n-1)$ is less than the area under the graph on the segment $1 \leqslant x \leqslant n$. This area equals $\ln n$, so $1 + 1/2 + 1/3 + \cdots + 1/n < \ln(n) + 1$.

b) On the same graph we see that $1/2 + 1/3 + \cdots + 1/(n-1)$ is greater than the area under the graph on segment $2 \leqslant x \leqslant n$, which equals $\ln(n) - \ln(2)$. So, $1 + 1/2 + 1/3 + \cdots + 1/(n-1) > \ln(n) - \ln(2) + 1 > \ln(n)$. Moreover, the difference between $1/2 + 1/3 + \cdots + 1/(n-1)$ and $\ln(n) - \ln(2)$ is growing with $n$.

The Lemma follows from combining a) and b). □

Note that Lemma 1 implies that $1 + 1/2 + 1/3 + \cdots = \infty$. The constant $\gamma$ is called *the Euler constant*, cf. Exercise 1.

Thus, for large $n$, formula (1) can be rewritten as $\ln(n) - \ln(k_0) \approx 1$, that is $\ln(n/k_0) \approx 1$. So, $k_0$ is approximately $n/e$.

In fact, one can show that for any $n$, $k_0$ is either integer part of $n/e$, $[n/e]$, or $[n/e] + 1$.

In the same way $C_{k_0}$ is approximately $1/e$. So, the chances of $A$ to win decline with $n$ from $1/2$ ($n = 2$ case) to $1/e$ ("$n = \infty$" case).

**Fact.** The strategy $St_{k_0}$ is indeed the best strategy of all possible strategies.

## 2. Stable marriage

2.1. **Simple stable marriage.** Consider $n$ men and $n$ women. We would like to marry all of them to each other, meaning to pair them up, so that in each pair there is a man and a woman. We will call such pairing *a matching*. There are $n!$ matchings.

Let $M$ be a matching. Suppose we have married couples $(m_1, w_1)$ and $(m_2, w_2)$. Suppose $m_1$ prefers $w_2$ to $w_1$ and $w_2$ prefers $m_1$ to $m_2$. In this case, the matching is not stable since $m_1$ and $w_2$ would leave their partners and marry each other. Such pair $(m_1, w_2)$ is called *a blocking pair*. A matching is called *stable* if there are no blocking pairs.

**Question.** Given lists of preferences, does there exist a stable matching?

**Examples.** a) Suppose all men and all women have the same list of preferences. Then there exists a unique stable matching – the best man marries the best woman, the second best man marries the second best woman, etc.

b) Suppose all men have the same preferences. Then there is a unique stable matching. Namely, the best woman marries her first preference, then the second best woman marries her first preference out of men left, then the third marries from men left, and so on.

c) Suppose any two men have different first preferences. Then there is at least one stable matching: all men marry their first choices. If, in addition, all women have different first preferences then there is another stable matching.

d) Let $n = 3$, let we have preferences:

| *men* | *women* |
| --- | --- |
| $1, 2, 3$ | $2, 3, 1$ |
| $3, 2, 1$ | $2, 1, 3$ |
| $2, 1, 3$ | $3, 1, 2$ |

That is the first man prefers the first woman to the second and the second to the third, the second man prefers the third women to the second and the second to the first, etc. There are 3 stable matchings: $\{(1, 1), (2, 3), (3, 2)\}$, $\{(1, 2), (2, 3), (3, 1)\}$, $\{(1, 3), (2, 2), (3, 1)\}$.

**The algorithm.**

Start with all men and women free.
While there is a single man $m$, do:
  Let $w$ be the best woman in $m$ list who $m$ did not propose to yet.
  Make $m$ propose to $w$.(but not before)
  If $w$ is free then $m$ and $w$ become engaged.
  If $w$ is engaged and likes her party better than $m$ then $m$ is rejected.
  If $w$ is engaged and likes $m$ better than her party then $w$ breaks up with her partner and gets engaged to $m$.
If all men are engaged, marry all the engaged couples.

**Theorem 2.** *The algorithm always stops at some point.*

*Proof*: Each man cannot propose more than $n$ times. If a man proposes the $n$th time (to the worst women on his list), it means that all other women have rejected him (or broke up with him for somebody else). Hence, all other women are engaged and the woman he proposes to is bound to be free. $\square$

**Theorem 3.** *The resulting matching is stable.*

*Proof*: Suppose $(m, w)$ is a blocking pair. It means that $m$ has proposed to $w$ and either was rejected or their engagement was broken. In both cases $w$ got a better partner then $m$. Notice that after a woman got engaged her next partners get only better. So, $(m, w)$ cannot be a blocking pair and we have a contradiction. $\square$

Note that we proved that the algorithm stops and produces a stable matching in no more than $n^2$ cycles. This is very fast: even to check that some given matching is stable one has to check $n^2$ possible blocking pairs!

**Corollary 4.** *For any list of preferences there is at least one stable matching.*

**Theorem 5.** *Any use of the algorithm (with men as proposers) yields the same result.*

*Proof*: Suppose we have a result of our algorithm, $M_1$, and another stable matching $M_2$. Suppose a man $m$ is married to $w_1$ in $M_1$ and to $w_2$ in $M_2$. We claim that $m$ prefers $w_1$ to $w_2$.

Suppose $m$ prefers $w_2$ to $w_1$. It means that $w_2$ has rejected $m$ at some point when the algorithm produced $M_1$. Thus some men were rejected by stable partners. Without loss of generality, assume that it happened the first time when $w_2$ rejected $m$.

Then the partner of $w_2$ in $M_1$ at the time when she rejected $m$, we call him $m_1$, can have no stable partners better than $w_2$. Then $(m_1, w_2)$ is a blocking pair in $M_2$. Contradiction.

Thus, in any matching we can get from our algorithm, each man obtains the best partner he can get in ANY stable matching. Obviously, such a matching is unique. □

**Theorem 6.** *In the resulting matching each man gets the best partner he can have in a stable matching, and each woman gets the worst partner she can get in a stable matching.*

*Proof*: We have already proved the first part. Now, let $M_0$ be the matching obtained from the algorithm. Let $M$ be some other stable matching. Suppose a woman $w$ likes her partner in $M_0$, $m_1$, more than her partner in $M_1$, $m_2$. Then the pair $(m_2, w)$ is blocking for $M_1$. □

**Corollary 7.** *Pair up each man with his best stable partner. Then it is a matching, moreover, this matching is stable.*

We will call the stable matching produced by the algorithm with man as proposers the men-oriented matching and denote it $M_0$.

Note that the stable matching is unique if and only if the men-oriented version and the women-oriented version of the algorithm give the same result.

**Theorem 8.** *Let $M_1, M_2$ be stable matchings. Let $m$ and $w$ be matched in $M_1$ but not in $M_2$. Then one of them prefers $M_1$ and another one prefers $M_2$.*

*Proof*: Denote $X_1$ the set of men who like $M_1$ better, $Y_1$ the set of women who like $M_1$ better. Also denote $X_2$ the set of men who like $M_2$ better, $Y_2$ the set of women who like $M_2$ better.

Then in $M_1$ there is no pair $(m, w)$, such that $m \in X_1$, $w \in Y_1$. Otherwise such pair would block $M_2$. In particular $|X_1| \leqslant |Y_2|$. Similarly, in $M_2$ there is no pair $(m, w)$, such that $m \in X_2$, $w \in Y_2$ because such pair would block $M_1$. In particular $|Y_2| \leqslant |X_1|$.

It means that $|Y_2| = |X_1|$ and all men from $X_1$ are matched to women in $Y_2$, and all men from $X_2$ are matched to women from $Y_1$. □

**Corollary 9.** *Let $M_1$, $M_2$ be stable matchings. Then the number of people prefering $M_1$ to $M_2$ equals the number of people prefering $M_2$ to $M_1$.*

Let $M_1, M_2$ be two stable matchings. We will say $M_1 \leqslant M_2$ if no man prefers $M_1$ to $M_2$. Then $M_2 \geqslant M_1$ if no woman prefers $M_2$ to $M_1$. Some remarkable properties of this ordering are described in Exercise 5.

2.2. **Some generalizations.** Suppose we have the same problem with less number of men than women. If we apply the same (men-oriented) version of the algorithm, we obtain a stable pairing. So, a stable pairing always exists. In fact, all Theorems 2-8 still hold true, cf. Exercise 8.

**Theorem 10.** *In all stable matchings the set of unmatched women is the same.*

*Proof*: Let $w$ be unmatched in some stable matching $M_1$ but matched in $M_0$ with $m$. Then $w$ is the best stable partner to $m$ (analogously to Theorem 6), and pair $(m, w)$ blocks $M_1$. □

Let us now allow people declare some partners totally unacceptable. We think that a person would rather stay single than marry an unacceptable partner. Again, in this case one can prove analogs of Theorems 2-8, cf. Exercise 9.

**Theorem 11.** *In all stable matchings the sets of married men and women are always the same.*

*Proof*: Let $M_1, M_2$ be two stable matchings. Let's construct a directed graph. Draw a vertex for each man and for each woman. Draw an edge from a man $m$ to a woman $w$ if $m$ and $w$ are married in $M_1$. Also draw an edge from a woman $w$ to her partner in $M_2$. So, in each vertex, at most one edge is coming in and at most one edge is coming out.

Suppose there is a man $m$ matched in $M_1$ but not matched in $M_2$. Then starting at $m$ there is a path. This path cannot have loops, so it is unique and must end. Each man on this path prefers $M_1$ (applying the analog of Theorem 8), so it cannot end on a man. Similarly, each woman on this path prefers $M_2$, so the path cannot end on a woman. Contradiction. □

Some other generalization are described in Exercises 11 and 12.

2.3. **False preferences.** Suppose some group of people found out the lists of preferences submitted by others. These people want to falsify their lists of preferences in order to obtain a better partner. Curiously, one can prove

**Theorem 12.** *Let $M_0$ be the men-optimal matching with honest preference lists. Then no coalition of people can falsify their preferences in such a way that all members of the coalition get a better partner in some stable matching based on false list of preferences than in $M_0$.*

Suppose we have a women oriented society, meaning the women-oriented version of algorithm is used.

**Theorem 13.** *Men can falsify their preferences in such a way, that women oriented version of the algorithm produces the same result as men oriented version of algorithm based on honest preferences.*

*Proof*: Men can declare all women except (worse then) their best stable partner unacceptable. □

## 3. Exercises

1. Recall that the Euler constant $\gamma$ is given by

$$\gamma = \lim_{n\to\infty}(1 + 1/2 + 1/3 + \cdots + 1/n - \ln n); \tag{2}$$

Use you calculator/computer to show that the Euler constant is $\gamma = 0.577216...$ . What should be $n$ in (2) to get the $k$th decimal digit of $\gamma$ correct?

2. Recall that the number $e$ is given by

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\frac{1}{n!} + .... \tag{3}$$

Use your calculator/computer to show that $e = 2.718281828...$ . What should be $n$ in (3) to get the $k$th decimal digit of $e$ correct? Compare to Exercise 1.

3. A person meets $n$ possible future spouses one by one in some order. S/he can either marry or reject the person s/he's just met; then meet another person. S/he can get married only once and s/he cannot marry a person which s/he has already rejected. We say s/he wins if s/he marries one of the best two choices out of all $n$ options. Find a strategy which gives more than fifty percent chance to succeed.

4. Let $F_n$ be the Fibonacci numbers, that is $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Prove that there exist lists of preferences for $n$ men and $n$ women, such that there are at least $F_n$ stable matchings.

5. Consider a problem of stable marriage.

a) Prove that for any two stable matchings $M_1$, $M_2$, there exists a unique stable matching $M$, such that $M_1 \leqslant M$, $M_2 \leqslant M$ and there is no stable matching $N$ such that $M_1 \leqslant N$, $M_2 \leqslant N$ and $N \leqslant M$. Denote $M = M_1 + M_2$.

b) Prove that for any two stable matchings $M_1$, $M_2$, there exists a unique stable matching $M$ such that $M_1 \geqslant M$, $M_2 \geqslant M$ and there is no stable matching $N$ such that, $M_1 \geqslant N$, $M_2 \geqslant N$ and $N \geqslant M$. Denote $M = M_1 * M_2$.

c) Let $M_1, M_2, M_3$ are stable matchings. Prove $M_1*(M_2+M_3) = (M_1*M_2)+(M_1*M_3)$.

d) Let $M_1, M_2, M_3$ are stable matchings. Prove $M_1+(M_2*M_3) = (M_1+M_2)*(M_1+M_3)$.

6. Let $n = 4$. In this case we have at most 10 stable matchings. The following lists or preferences allow exactly 10 stable matchings:

| *men* | *women* |
|---|---|
| $1, 2, 3, 4$ | $4, 3, 2, 1$ |
| $2, 1, 4, 3$ | $3, 4, 1, 2$ |
| $3, 4, 1, 2$ | $2, 1, 4, 3$ |
| $4, 3, 2, 1$ | $1, 2, 3, 4$ |

Find all 10 stable matchings and compute products and sums of all stable matchings.

7. Prove Theorem 12.

8. Prove Theorems 2-8 when numbers of men and women are different.

9. Prove Theorems 2-8 when unacceptable partners are allowed.

10. Consider **the** following **stupid algorithm**. It starts at a random matching. Given a matching, the stupid algorithm finds some blocking pair $(m_1, w_2)$ (say, $m_1$ is matched to $w_1$ and $w_2$ is matched to $m_2$), and forms a new matching which has all the same pairs except for $(m_1, w_1)$and $(m_2, w_2)$ which are changed to $(m_1, w_2)$, $(m_2, w_1)$. It stops if the matching is stable. Is it true that the stupid algorithm always stops?

11. **The hospital/residents problem.** Suppose there are several hospitals and several residents. Each hospital can allocate some number of residents (may be more than one and different hospitals can have different sizes). Both residents and hospitals have lists of preferences. The definition of stable matching is the same as in the stable marriage problem. Formulate and prove Theorems 2-13 adjusted to this setting.

12. **The roommate problem.** Suppose we have $2n$ people which we want to allocate to $n$ rooms (each room is for two people). Each person has a list of preferences. The stable matching is defined as in the stable marriage problem. Find an example when no stable matching is possible.

Please email your **remarks and questions** to **mukhin@msri.org** .