Problems for Approximately Rational Numbers by Dmitry Fuchs

1. Let $q = 2^m 5^n r, a = max(m, n)$ and b the minimal number of 9's such that $\underbrace{999...9}_{b}$ is divisible by r. Then in decimal representation, the fraction $\frac{p}{q}$ has period of length b, and a initial decimal digits before its period starts.

2. (a) For every irrational α there exists infinitely many $\frac{p_k}{q_k}$ such that
$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{\sqrt{5}q^2}.$$
   (b) There are irrational numbers, α such that for every $\lambda > \sqrt{5}$ there exists finitely many $\frac{p}{q}$'s such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda q^2}.$$
   (c) If $\alpha \neq$ the continued fraction $[a_0, a_1, a_2, ...]$ where all but finitely many of the partial quotients, a_k's, are not 1's then the $\sqrt{5}$ in part (a) can be replaced with $\sqrt{8}$

3. Given two vectors, $\mathbf{a}$ and $\mathbf{b}$, from the origin, let a_0 be the largest integral multiple of $\mathbf{b}$ that can be "added" to the tip of $\mathbf{a}$ without crossing the y-axis. Let $\mathbf{v_1}$ be the vector $\mathbf{a} + a_0\mathbf{b}$. a_1 is the largest integral multiple of $\mathbf{v_1}$ that can be "added" to $\mathbf{b}$ without crossing the y-axis. Let $\mathbf{v_2}$ be the vector $\mathbf{b} + a_1\mathbf{a}$. a_2 is the largest integral multiple of $\mathbf{v_1}$ that can be "added" to $\mathbf{a}$ without crossing the y-axis. Continuing in this way the sequence $a_0, a_1, a_2, ...$ is formed. Show that $\frac{\|\mathbf{a}\|}{\|\mathbf{b}\|}$ equals the continued fraction $[a_0, a_1, a_2, ...]$