Berkeley Math CircleCombinatorics ISept. 20, 1998

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Some of these problems we will discuss today, and the rest are homework, although I warn you that the problems involve many topics in combinatorics, some of which we will not discuss until November 8. They are arranged in roughly increasing order of difficulty.

1 An n-bit string is n-digit "word," each digit of which is either 0 or 1. How many n-bit strings contain at least 1 zero?

2 Imagine a piece of graph paper. Starting at the origin draw a path to the point $(10, 10)$, that stays on the grid lines (which are one unit apart) and has a total length of 20. For example, one path is to go from $(0, 0)$ to $(0, 7)$ to $(4, 7)$ to $(4, 10)$ to $(10, 10)$. Another path goes from $(0, 0)$ to $(10, 0)$ to $(10, 10)$. How many possible different paths are there?

3 How many different ordered triples (a, b, c) of non-negative integers are there such that $a + b + c = 50$? What if the three integers had to be positive?

4 Define a **domino** to be a 1×2 rectangle. In how many ways can a $n \times 2$ rectangle be tiled by dominos?

5 Suppose we have three different toys and we want to give them away to two girls and one boy (one toy per child). The children will be selected from 4 boys and 6 girls. In how many ways can this be done?

6 Suppose again that we have three different toys and we want to give them away (one toy per kid) to three children selected from a pool of 4 boys and 6 girls, but now we require that at least two boys get a toy. In how many ways can this be done?

7 How many ways can the positive integer n can be written as an ordered sum of at least one positive integer? For example,

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1,$$

so when $n = 4$, there are 8 such *ordered partitions*.

8 Ten children order ice cream cones at a store featuring 31 flavors. How many orders are possible in which at least two children get the same flavor?

9 In how many ways can two squares be selected from an 8-by-8 chessboard so that they are not in the same row or the same column?

10 In how many ways can four squares, not all in the same row or column, be selected from an 8-by-8 chessboard to form a rectangle?

11 In how many ways can we place r red balls and w white balls in n boxes so that each box contains at least one ball of each color?

12 There are 10 adjacent parking spaces in the parking lot. When you arrive in your new Rolls Royce, there are already seven cars in the lot. What is the probability that you can find two adjacent unocupied spaces for your Rolls? Generalize.

13 Find the maximum number of regions in the plane that are determined by n "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

14 How many ways can you tile a $3 \times n$ rectangle with 2×1 dominoes?

15 How many subsets of the set $\{1, 2, 3, 4, \ldots, 30\}$ have the property that the sum of the elements of the subset is greater than 232?

16 Prove that for all positive integers n,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n2^{n-1}.$$

17 Given n points arranged around a circle and the chords connecting each pair of points is drawn. If no three chords meet in a point, how many points of intersection are there? For example, when $n = 6$, there are 15 intersections.

18 Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$.

19 How many five-card hands from a standard deck of cards contain at least one card in each suit?

20 Four young couples are sitting in a row. In how many ways can we seat them so that no person sits next to their "significant other?"

21 Sal the Magician asks you to pick any five cards from a standard deck.[1] You do so, and then show them to Sal's assistant Pat, who places one of the five cards back in the deck and then puts the remaining four cards into a pile. Sal is blindfolded, and does not witness any of this. Then Sal takes off the blindfold, takes the pile of 4 cards, reads the four cards that Pat has arranged, and is able to find the fifth card in the deck (even if you shuffle the deck after Pat puts the card in the deck). Assume that neither Sal nor Pat have supernatural powers, and that the deck of cards is not marked. How is the trick done? Harder version: you pick the which of the five cards goes back into the deck (instead of Pat)

22 Eight people are in a room. One or more of them get an ice-cream cone. One or more of them get a chocolate-chip cookie. In how many different ways can this happen, given that at least one person gets both an ice-cream cone and a chocolate-chip cookie?

23 How many *strictly increasing* sequences of positive integers begin with 1 and end with 1000?

24 For any set, prove that the number of its subsets with an even number of elements is equal to the number of subsets with an odd number of elements. For example, the set $\{a, b, c\}$ in the problem above has 4 subsets with an even number of elements (the empty set has 0 elements, which is even), and 4 with an odd number of elements.

25 Find a nice formula for the sum

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Can you explain why your formula is true?

26 Ten dogs encounter 8 biscuits. Dogs do not share biscuits! How many different ways can the biscuits can be consumed

(a) if we assume that the dogs are distinguishable, but the biscuits are not;

(b) if we assume that the dogs and the biscuits are distinguishable (for example, each biscuit is a different flavor).

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[1]A standard deck contains 52 cards, 13 denominations $(2, 3, \ldots, 10, J, Q, K, A)$ in each of four suits $(\diamondsuit, \heartsuit, \clubsuit, \spadesuit)$.

(c) if we assume that neither the dogs nor the biscuits are distinguishable? (We are able to distinguish dogs from biscuits, however. The answer is *not* 1!)

27 Let S be a set with n elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

28 (USAMO 72) A random number generator randomly generates the integers $1, 2, \ldots, 9$ with equal probability. Find the probability that after n numbers are generated, the product is a multiple of 10.

29 Use a combinatorial argument to show that for all positive integers n, m, k with n and m greater than or equal to k,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}.$$

This is known as the **Vandermonde Convolution Formula**.

30 (Putnam 1996) Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

31 Let $u(n)$ denote the number of ways in which a set of n elements can be partitioned. For example, $u(3) = 5$, corresponding to

$$\{a, b, c\}; \quad \{a, b\}, \{c\}; \quad \{a\}, \{b, c\}; \quad \{a, c\}, \{b\}; \quad \{a\}, \{b\}, \{c\}.$$

Find a recurrence relation for $u(n)$. You might hope that $u(4) = 14$, suggesting a Catalan-style recurrence, but unfortunately, $u(4) = 15$.

32 (AIME 1988) In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order $1, 2, 3, 4, 5, 6, 7, 8, 9$.

While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after-lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.)

33 Imagine that you are going to give n kids ice cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavors get mixed, show that the number of ways we can give out the cones *using all k flavors* is

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots + (-1)^k \binom{k}{k} \cdots 0^n.$$

34 (IMO 89) Let a permutatation π of $\{1, 2, \ldots, 2n\}$ have property P if

$$|\pi(i) - \pi(i+1)| = n$$

for at least one $i \in [2n-1]$. Show that, for each n, there are more permutations with property P than without it.

35 Define $p(n)$ to be the number of different ways a positive integer n can be written as a sum of positive integers, where the order of the summands doesn't matter. Here is a table of the first few values of $p(n)$.

n	$p(n)$	The different sums
1	1	1
2	2	$1+1, 2$
3	3	$1+1+1, 1+2, 3$
4	4	$1+1+1+1, 1+1+2, 1+3, 4$
5	7	$1+1+1+1+1, 1+1+1+2,$
		$1+1+3, 1+2+2, 2+3, 1+4, 5$

Show that $p(n) \geq 2^{\lfloor \sqrt{n} \rfloor}$ for all $n \geq 2$.

36 Decide whether there exist 10,000 ten-digit numbers divisible by seven, all of which can be obtained from one another by a reordering of their digits.

37 (USAMO 1996) Let a_n be the number of n-bit strings containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of n-bit strings that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

38 (Putnam 1993) Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \rightarrow \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$