COMPLEX NUMBERS AND GEOMETRY
BERKELEY MATH CIRCLE

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Complex numbers were discovered in order to solve polynomial equations. If we introduce $i = \sqrt{-1}$, then any complex number can be written in the form $z = a + bi$, where a and b are real numbers. The sum and the product of complex numbers are defined as

$$(a + bi) + (c + di) = (a + b) + (c + d)\,i,$$

$$(a + bi)\,(c + di) = (ac - bd) + (ad + bc)\,i.$$

We discuss today some applications of complex numbers to geometry. One can think about complex number $z = a+bi$ as a vector on the plane whose x-coordinate is a and y-coordinate is b. Then the addition (subtraction) of complex numbers is the same as the addition (subtraction) of vectors. To understand multiplication geometrically we define the *argument* $\alpha = \arg z$ of a complex number z as the counterclockwise angle of the vector z with x-axis. For example, $\arg i = 90°$, $\arg(i - 1) = 135°$, $\arg(1 - i) = -45°$. The *absolute value* $|z|$ of a complex number z is by definition the length of the vector

$$|z| = \sqrt{a^2 + b^2}.$$

For example, $|i| = 1$ and $|i - 1| = \sqrt{2}$.

Problem 1. Show that to multiply two complex numbers one has to add arguments and multiply absolute values, that is

$$|zw| = |z||w|, \ \arg zw = \arg z + \arg w.$$

Problem 2. Prove that the equation $z^n = 1$ has exactly n complex solutions and draw them all on the complex plane.

Problem 3. How many times during the day do the minute and hour hands on the clock face coincide?

Problem 4. (Strange clock). This clock's minute and hour hands have the same length. How many times during the day you can not use this clock to check time?

Complex conjugation. If $z = a+bi$ is a complex number, then the conjugate number is defined as $\bar{z} = a - bi$.

Problem 5. Show that the complex conjugation is the reflection with respect to x-axis in the complex plane. Check that

$$\overline{z + w} = \bar{z} + \bar{w}, \ \overline{zw} = \bar{z}\bar{w}, \ \arg \bar{z} = -\arg z, \ |\bar{z}| = |z|, \ z\bar{z} = |z|^2.$$

Using properties of complex conjugation we can divide by any complex number except 0 using the rule

$$\frac{z}{w} = \frac{1}{|w|^2} z\bar{w}.$$

Problem 6. The pirates hid a treasure chest under the ground on a small Caribbean island. You stole their map with instructions. The instructions say, "Start from the gallows, go to the oak tree, turn right 90°, walk the same distance as from the gallows to the oak tree, put a sword at the point you stop. Return to the gallows, go to the birch tree, turn left 90°, then walk the same distance as from the gallows to the birch. Put another sword at the point where you stop. Dig at the midpoint between the two swords." You came to the island. The birch tree and the oak tree are there, but, alas, no gallows! Can you find the treasure?

Problem 7. Prove that three distinct points z_1, z_2 and z_3 lie on a line if and only if $\frac{z_1 - z_3}{z_2 - z_3}$ is real. Prove that four distinct points z_1, z_2, z_3 and z_4 lie on a circle if and only if

$$\gamma(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} \div \frac{z_1 - z_4}{z_2 - z_4}$$

is real.

The number $\gamma(z_1, z_2, z_3, z_4)$ is called the *cross-ratio* of four numbers z_1, z_2, z_3 and z_4.

Problem 8. Check that a transformation F of the complex plane defined by the formula

$$F(z) = Az + B,$$

where A and B are complex numbers maps a line to a line and preserves angles. If, in addition, $|A| = 1$, then F preserves distances. If $A \neq 1$, then F has a unique fixed point and F is a composition of a rotation and a dilation with centers at the fixed point.

Problem 9. Let G be a transformation of the complex plane which maps a line to a line and preserves angles. Then either G is as in Problem 8 or G is a composition of some F as in problem 8 and the complex conjugation.

Problem 10. Use problem 8 to prove that a composition of two rotations (centers may be different) is either a rotation or a parallel translation.

Define now a transformation of the complex plane by a formula

$$F(z) = \frac{Az + B}{Cz + D}$$

for some complex numbers A, B, C, D such that $AD - BC \neq 0$. It is not defined at the point $\frac{-D}{C}$. To define it everywhere consider one more point ∞ and put

$$F(\infty) = \frac{A}{C},\ F\left(\frac{-D}{C}\right) = \infty.$$

Problem 11. Prove that F is a map of the complex plane with ∞ onto itself. Find the formula for the inverse map.

A transformation F of extended complex plane defined above is called a *linear fractional transformation.*

Problem 12. Check that a linear fractional transformation preserves the cross-ratio, more precisely

$$\gamma\left(F\left(z_1\right), F\left(z_2\right), F\left(z_3\right), F\left(z_4\right)\right) = \gamma\left(z_1, z_2, z_3, z_4\right)$$

for any complex numbers z_1, z_2, z_3, z_4 and a linear fractional F.

Use this property to show that a linear fractional transformation maps any line to a circle or a line, and any circle to a circle or a line.

Problem 13. *Inversion* with center O and radius R is a map of the extended plane to itself which maps a point X to the point X' lying on the ray OX such that

$$|OX||OX'| = R^2.$$

In addition, O goes to ∞, and ∞ goes to O. Let O be the origin. Prove that the inversion can be defined by the formula

$$F\left(z\right) = \frac{R^2}{\bar{z}}.$$

Using this formula prove that an inversion maps any line to a circle or a line, and any circle to a circle or a line.

Problem 14. Check that the transformation

$$F\left(z\right) = \frac{iz - 1}{z - i}$$

maps the unit circle to the real axis (extended by ∞). A point z on the unit circle has rational coordinates if and only if $F\left(z\right)$ has a rational coordinate on the real line.

Problem 15. Use the previous problem to find all Pythagorean triples, which are integers (a, b, c) such that $c^2 = a^2 + b^2$. Hint: first look for rational solutions.

Inversion is very useful for straightedge and compass constructions.

Problem 16. Given a circle with center O and radius R and a point X, construct the image of X under the inversion with center O and radius R using straightedge and compass.

Problem 17. Given a point P and two circles C_1 and C_2, construct a circle passing through P and tangent to C_1 and C_2.

Problem 18. (Apollonius problem) Given three circles, construct a circle tangent to these three circles.

Problem 19. Suppose that your straightedge is broken. Any construction which can be performed using straightedge and compass can be done using compass only. We assume that a line is "constructed" if we have constructed two distinct points on it.

Spherical Geometry and stereographic projection.

We know that we do not live in a plane; assume that we live on a sphere of a large radius. We do not notice the difference if we do not move very far from home. Let us consider geometry on a sphere. To define a line, we should recall that a line gives the shortest path between two points. If you fly from San Francisco to Tokyo, what is the shortest path? Given two points P and Q on a sphere, define the distance $d(P,Q)$ between them as the length of the shortest arc of a big circle through P and Q. (By a big circle we mean the circle whose center coincides with the center of the sphere. If P and Q are not opposite to each other, there is exactly one big circle through them.)

Problem 20. Check the triangle inequality

$$d(P,Q) \leq d(P,R) + d(R,Q)$$

for any three points on the sphere. Use it to prove that a shortest path between two points on the sphere is given by an arc of a big circle through them.

Problem 21. Any transformation of a sphere which preserves distance is a rotation or a reflection in a big circle.

Two triangles on a sphere are *congruent* if there is a rotation or a reflection which moves one triangle to another.

Problem 22. Prove that two triangles are congruent if and only if they have the same angles.

Problem 23. The sum of angles of any triangle on a sphere is greater than $180°$.

Problem 24. Three lines in general position divide the plane in 7 parts. In how many parts do three big circles divide a sphere?

Problem 25. Consider a triangle on a sphere with angles α, β and γ. Let s denote the area of the triangle, assume that the area of the whole sphere is 1. Prove the formula

$$2s = \frac{\alpha + \beta + \gamma - 180°}{360°}.$$

Problem 26. In the plane geometry the angle bisectors of a triangle meet at one point. Is it true on a sphere? The same question for medians and altitudes of a triangle.

Let O denote the North pole of a sphere S and Π be the plane containing the equator. For each point P on the sphere let P' be the point of intersection of the line OP and the plane Π. The map $f(P) = P'$ maps the sphere S (without North pole) to Π. This map is called a *stereographic projection*.

Problem 27. Check that

$$|OP||OP'| = 2R^2,$$

where R is the radius of the sphere S. Consider the inversion in 3-dimensional space with center O and radius $\sqrt{2}R$. Check that this inversion maps a sphere not passing through O to a sphere, and a sphere passing through O to a plane. In particular, S goes to Π.

Problem 28. Check that the stereographic projection maps a circle on S to a circle or a line on Π, and that a big circle goes to a circle (line) which intersects the equator at two opposite points.

Problem 29. Check that a reflection on S corresponds to an inversion on Π and a rotation corresponds to a linear fractional map

$$F(z) = \frac{Az + B}{-\bar{B}z + \bar{A}}.$$

Here we assume that the radius of S is 1.