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AFFINE AND PROJECTIVE TRANSFORMATIONS OF THE PLANE.

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Problem 1. Given a trapezoid $ABCD$ such that the sides AD and BC are parallel. Prove that the point of intersection of AC and BD, the point of intersection of AB and CD and the midpoints of AB and CD lie on the same line.

One can see that the problem is very simple if the trapezoid is symmetric.

Problem 2. Show that the medians of a triangle meet at one point.

Problem 3. From each vertex of a triangle draw the lines which divide the opposite side in three equal parts. Show that the diagonals of the hexagon made by these lines meet at one point.

These problems are quite simple in case when a triangle is equilateral.

An *affine transformation* of the plane is a map $A \mapsto A'$ such that

(1) If $A \neq B$, then $A' \neq B'$;
(2) For any point X there exists a point Y such that $X = Y'$;
(3) Every line l goes to a line l'.

Problem 4. Show that an affine transformation maps parallel lines to parallel lines.

A transformation of a plane is called a *rigid motion* if it preserves the distances, i.e. $|AB| = |A'B'|$.

Problem 5. Show that a rigid motion is an affine transformation.

Problem 6. Suppose that a transformation is a dialation or shrinking, i.e. there is a positive number k such that $|AB| = k|A'B'|$. Then this transformation is affine.

An affine transformation may change distances but the following properties hold.

Problem 7. If AB and CD are parallel and $|AB| = |CD|$, then $|A'B'| = |C'D'|$.

Problem 8. If A, B and C lie on the same and $\frac{|AB|}{|CD|} = \frac{p}{q}$, then $\frac{|A'B'|}{|C'D'|} = \frac{p}{q}$.

Affine transformation in coordinates. Introduce the coordinates on the plane. One can define any transformation by a formula, for example the symmetry about the vertical axes can be defined by

$$x' = -x,\ y' = y,$$

and a parallel translation on a vector with coordinates (a, b) is given by the formula

$$x' = x + a,\ y' = x + b.$$

Problem 9. Write formulas for rotations on $180°, 90°$ and $45°$ about the origin.

Consider the transformation given by the formula

$$x' = ax + by + e,\ y' = cx + dy + f, \tag{0.1}$$

for some numbers a, b, c, d, e, f.

Problem 10. Show that if $ad-bc \neq 0$, then (0.1) defines an affine transformation. Any affine transformation is given by (0.1) for suitable a, b, c, d, e and f.

Problem 11. Check that for any two triangles there exists an affine transformation which maps one triangle to another. Use this result to solve problems 1,2 and 3.

Problem 12. What happens with circles after affine transformation?

Central projection. Consider two planes Π and Π' in three-dimensional space. Let O be a point that does not belong to Π and Π'. For any X in Π draw the line OX and let X' be the point where OX meets Π'.

Suppose that Π and Π' are not parallel. One can see that X' is defined for any X which does not belong to a special line l. This special line is the line of intersection of Π and the plane Π'' parallel to Π' and passing through O. In the same way the plane Π' has an exceptional line m. For any point Y in Π' which does not belong to m, one can find X in Π such that $Y = X'$. Therefore we constructed a map from Π with removed l onto Π' with removed m. This map is called a *central projection*.

Problem 13. Check that a central projection maps lines of Π to the lines on Π'. If a and b are two parallel lines in Π, then a' and b' are not parallel! They meet at some point of the exceptional line m in Π'.

In order to avoid the problem of exceptional lines, we introduce a notion of *projective plane.* A projective plane is obtained from the usual plane by adding some points, called *points at infinity.* There is one point at infinity for each direction on the plane. Taken together points at infinity form *the line at infinity.* On projective plane parallel lines are not special. We assume that all parallel lines in the same direction meet at one point at infinity. Thus any two lines on projective plane meet at one point, and there exists exactly one line passing through two points even if one or both points lie at infinity.

A *projective transformation* is a transformation of projective plane which maps lines to lines. A central projection is a projective transformation which maps l to the infinity of Π' and the line at infinity of Π to the line m.

Problem 14. Check that a projective transformation which maps the line at infinity to itself is affine.

Problem 15. Let Γ be a circle, and B be a point inside Γ. Show that there exists a projective transformation which maps Γ to itself and B to the center of Γ.

Problem 16. Let A, B, C, D be points on projective plane such that any three points do not lie on the same line. Let A', B', C' and D' satisfy the same property. Show that there is a projective transformation which moves A to A', B to B', C to C' and D to D'.

Problem 17. Assume that a quadrilateral is circumscribed about a circle. Show that the diagonals of the quadrilateral and the lines joining the opposite points of tangency meet at one point.

Problem 18. (Brianchon's theorem) Let $ABCDEF$ be a hexagon circumscribed about a circle. Show that AD, BE and CF meet at one point.

Problem 19. (Pascal's theorem) Let A, B, C, D, E and F lie on the same circle. Let AB and DE meet at X, BC and EF meet at Y, CD and AF meet at Z. Prove that X, Y and Z lie on the same line.

Problem 20. Show that one can not construct the center of a circle drawn on the plane using a ruler only.