Archimedes and the Arbelos

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1 History and Sources

(This talk was originally given in January of 2000.) Archimedes lived from 287 BC until he was killed by a Roman soldier in 212 BC. He is usually considered to be one of the three greatest mathematicians of all time, the other two being Newton and Gauss. The extant works of Archimedes are readily available today in Heath [2] and Dijksterhuis [3]. Both contain the works with extensive notes and historical information. They both can be somewhat difficult to plow through at times. To ease the burden, a new book by Sherman Stein [3] came out in 1999 that is accessible to a much broader audience. It fact the only prerequisite is high school algebra and geometry. A partial listing of the works that have not been lost is as follows : *On the Equilibrium of Planes, Quadrature of the Parabola, On the Sphere and Cylinder, On Spirals, On Conoids and Spheroids, On Floating Bodies, Measurement of a Circle, The Sand-reckoner, and The Book of Lemmas.*

2 The Arbelos ($\acute{\alpha}\rho\beta\eta\lambda o\varsigma$)

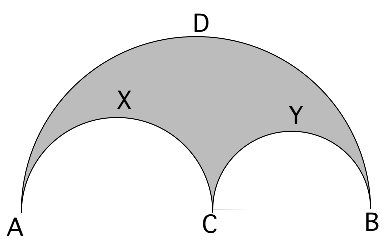

Figure 1

The arbelos consists of three points A,B,C which are collinear, together with the semicircles ADB, AXC and CYB as shown in Figure 1. It was so named because of its shape, which resembles a shoemaker's knife, or $\acute{\alpha}\rho\beta\eta\lambda o\varsigma$. It engaged the attention of no less a mathematician than Archimedes. He played with this figure for fun, which is an excellent reason for doing mathematics. His theorems about the figure are contained in the Book of Lemmas which come to us in the form of an Arabic manuscript that details what Archimedes proved.

For fun and relaxation, try proving the following statements. In the figure $\overline{CD}$ has been added to the figure tangent to the two small semicircles. $\overline{AD}$ intersects a small semicircle at X and $\overline{BD}$ intersects the other small semicircle at Y. $\overline{XY}$ intersects $\overline{AD}$ at P.

1. The area of the arbelos is equal to the area of the circle with diameter $\overline{CD}$.

2. $\overline{XY}$ and $\overline{CD}$ bisect each other.

3. $\overline{XY}$ is tangent to the small semicircles.

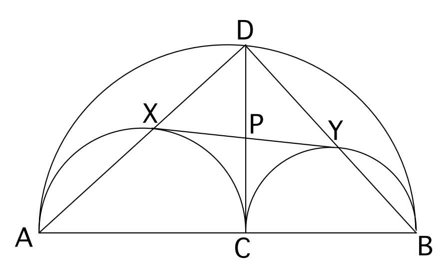

Figure 2

4. If two circles are tangent at A, and if $\overline{BD}$, $\overline{EF}$ are parallel diameters in the circles, then A, D, and F are collinear. (This is Proposition 1 in *The Book of Lemmas*).

5. The two semicircles inscribed in regions ACD and BCD have equal radii. See Figure 3. Find this radius in terms of the radii of the three semicircles that form the arbelos. (This is Proposition 5 in *The Book of Lemmas*).

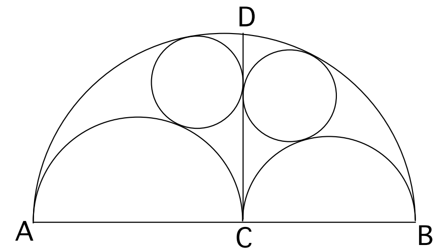

Figure 3

6. Construct with proof, the twin circles in a given arbelos with a straightedge and compass.

7. Find the radius of the circle tangent to all three of the semicircles that form the arbelos (See Figure 4) in terms of the radii of these three semicircles. (This is Proposition 6 in *The Book of Lemmas*). It was the basis for the following problem at the **East Bay Mathletes Competition, April 1999**.

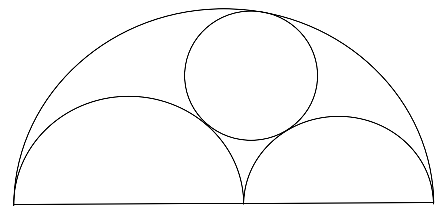
Figure 4

8. In Figure 4, the small circle is tangent to all three of the semicircles. If the diameters of the two small semicircles are 10 and 6 units, then what is the radius of this small circle? Answer as a fraction in lowest terms. (Hint: It can be easily proved by inversion that the distance of the center of the small circle from the base is equal to the diameter of the small circle.) See the next problem for more on this and problem 4 in the next section for a generalization.

9. (**Pappus**) The chain of inscribed circles, C_n, in an arbelos has the following property. The distance from the diameter of the largest semicircle to the center of the nth circle in the chain, C_n, in the chain, C_n, is exactly equal to d_n, where d_n is the diameter of C_n. (Hint: Use inversion in a circle orthogonal to C_n.)

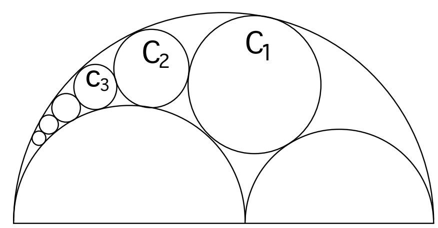

Figure 5

10. The centers of the chain of inscribed circles in an arbelos lie on an ellipse with foci at the centers of the two semicircles to which each circle of the chain is tangent.

For a discussion of how Pappus proved this result in problem 9, (The technique of inversion was not known until the 19th Century.) see the article *How Did Pappus Do It?* by Leon Bankoff on pages 112-118 of the book *The Mathematical Gardner* edited by David Klarner and published by Pridle, Weber & Schmidt in 1981. The book was written as a tribute to Martin Gardner. Martin Gardner also wrote a column on the arbelos for *Scientific American* in January 1979. The column is also available in *Fractal Music, Hypercards and More...* which reprints his columns from 1978 and 1979.

There have been a number of developments since I last gave this talk. A teacher from Spain contacted me and asked for some of the references in my paper, which I sent to him. A year later he sent me a pdf file of a 47 page paper (in Spanish) he produced on the arbelos, which included the proofs of most of the problems from this paper and more [**12**]. Thomas Schoch [**11**] contacted me to tell me that I had misspelled his name in the references. This year he informed me about his amazing site on the web [**13**] where you can read his story, see beautiful interactive animations of everything in this talk and much more, including the Schoch line and Woo circles. To cap it off, a month ago, the March 2006 issue of *The American Mathematical Monthly* came out with a 13 page reflection on the arbelos.[**14**]

3 Some Generalizations

There are many generalizations of the ideas presented in the preceding section. The following are a few with which one can play. In addition, the references point to many directions of fruitful research for anyone who wishes to pursue the topic in more depth.

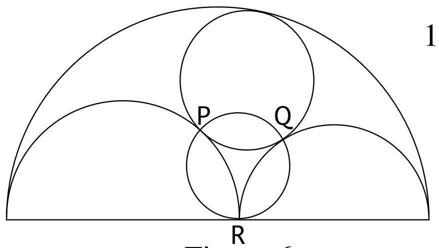

Figure 6

1. (**The twins are triplets**) Let P and Q be the points where the circle inscribed in the arbelos is tangent to the two smaller semicircles, and let R be the point where the two smaller semicircles are tangent as in Figure 6. The circle containing P, Q and R has a radius equal to that of the twin circles, namely $r_1 r_2/r$. See [**8**].

2. (**Quadruplets**) The largest circle inscribed in the circular segment of the largest semicircle formed by the chord $\overline{EF}$ tangent to the other two semicircles also has the same radius as the twins. See Figure 7. Note that the point of tangency for this circle is the point D of the common internal tangent to the smaller semicircles of the arbelos, $\overline{CD}$. The story doesn't end here. There are infinite families of circles in and around the arbelos with the same radius as the twins. See [**10**] and [**11**].

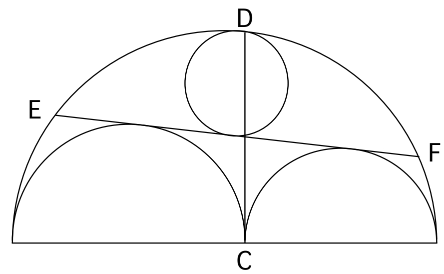

Figure 7

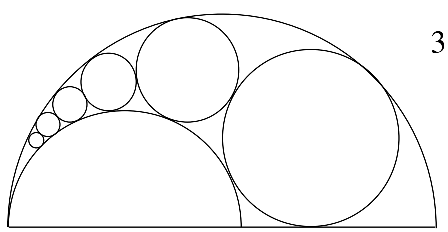
Figure 8

3. A chain of circles, inscribed in two internally tangent semicircles with collinear endpoints, begins with a circle tangent to the line of centers of the semicircles. The distance of the center of the nth circle in the chain from the line of centers is $(2n-1)r_n$ where r_n is the radius of the nth circle.

4. In Figure 5, if the larger semicircle on the base has a radius k times the radius of the smaller semicircle, where k is an integer, and the chain of circles, $C_1, C_2, C_3, \ldots$ is formed, then the centers of largest (outer) semicircle, the smallest semicircle and the centers of C_k and C_{k+1} form a rectangle. See *On a Generalization of the Arbelos* by M. G. Gaba in the *American Mathematical Monthly*, vol 47, Jan. 1940, pp 19-24.

5. (**Valentine**) Let r, r_1, r_2 be the radii of the semicircles forming an arbelos and let ρ be the radius of the inscribed circle. The ratio of the area of the arbelos to the area of the heart equals $\dfrac{\rho}{r} = \dfrac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2}$. See [**9**] for ten proofs by Charles W. Trigg. See [**4**] for seven proofs that $\rho = \dfrac{r r_1 r_2}{r_1^2 + r_1 r_2 + r_2^2}$.

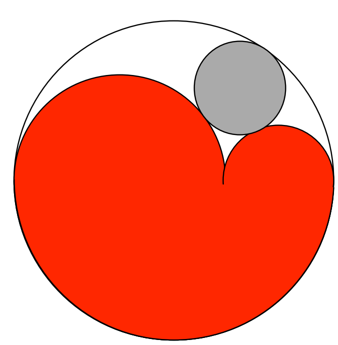
Figure 9

4 References

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2. T.L. Heath, *The Works of Archimedes* .Dover Pubilications,Inc. Reissue of the 1897 edition with the 1912 Supplement on *The Method*.

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5. Brother L. Raphael,F.S.C. *The Shoemaker's Knife*, *Mathematics Teacher*, April 1973.

6. Ogilvy, C. S. *Excursions in Geometry* Oxford University Press, 1969

7. Harold Jacobs. *Geometry*. First Edition page 534. W.H.Freeman and Company, 1974.

8. Leon Bankoff. *Are the Twin Circles of Archimedes really Twins?*. pp 214-218 Vol 47 (1974) *Mathematics Magazine*, Mathematical Association of America.

9. Charles W. Trigg. *How Do I Love Thee? Let Me Count the Ways*. *Eureka* now *Crux Mathematicorum*, pp 217-224 Vol. 3 No. 8 October 1977.

10. Leon Bankoff. *The Marvelous Arbelos*. pp 247-253 *The Lighter Side of Mathematics*, Mathematical Association of America, 1994.

11. Clayton W. Dodge, Thomas Schoch, Peter Y. Woo, Paul Yiu *Those Ubiquitous Archimedean Circles*. pp 202-213 Vol 72 No 3 June 1999 *Mathematics Magazine*, Mathematical Association of America.

12. Francisco Javier Garcia Capitán, *Sobre el árbelos* . Private Communication, 2002.

13. Thomas Schoch, *Arbelos: Amazing Properties*, http://www.retas.de/thomas/arbelos/arbelos.html

14. Harold Boas, *Reflections on the Arbelos*, pp 236-249 March 2006 *The American Mathematical Monthly*, Mathematical Association of America. There is a preprint available on the internet at: http://www.math.tamu.edu/ harold.boas/preprints/arbelos.pdf

    In 1998, a palimpsest of Archimedes work discovered in 1907, resurfaced after disappearing for most of the twentieth century. The following scholarly work one of the fruits of this find.

15. Reviel Netz, *The Works of Archimedes* Volume 1: The Two Books *On the Sphere and the Cylinder*, Cambridge University Press, 2004.

    In another direction that explores packing circles with circles, go to the following article on the *Science News Online* website at: http://www.sciencenews.org/articles/20010421/bob18.asp

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us