Generating functions.

Definition. Let a_o, a_1, ... (denoted also by $(a_n)_{n=0}^{\infty}$) be a sequence of interest. The formal power series

$$G(x) := \sum_{n=0}^{\infty} a_n x^n$$

is called the *ordinary generating function* for the sequence (a_n).

The coefficient of x^n in $G(x)$ is a_n:

$$a_n = [x^n]G(x).$$

Example 1. How many ways are there to split a dollar bill into pennies, nickels, dimes, and quarters?

Example 2. Let b_n denote the number of representations of a positive integer n as a sum of numbers 1, 2, 3, and 4, where two sums with different orders of summands are considered different. What is $\sum_{n=0}^{\infty} b_n x^n$?

Example 3. Show that the generating function for the Fibonacci sequence (F_n) is

$$G(x) = \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-ax} - \frac{1}{1-bx}\right)$$

where $a := (1+\sqrt{5})/2$, $b := (1-\sqrt{5})/2$. Thus $F_n = (a^n - b^n)/\sqrt{5}$.

Example 4. The number of partitions of $[n] := \{1, 2, \ldots, n\}$ into k nonempty blocks is the *Stirling number of the second kind* denoted by $\left\{ {n \atop k} \right\}$. Find the generating function for $\left\{ {n \atop k} \right\}$.

Example 5. The number of permutations of $[n]$ with k cycles is the *Stirling number of the first kind* denoted by $\left[{n \atop k} \right]$. Find the generating function for $\left[{n \atop k} \right]$.

Definition. Let S be a set and let w be a function from S to $\mathbb{Z}_+$. We refer to $w(\sigma)$ as the *weight* of σ. The generating function for S with respect to w is defined by

$$G_S(x) := \sum_{\sigma \in S} x^{w(\sigma)}.$$

Example 6. Let S be the set of permutations of [3] and let w be the function counting the number of fixed points. Find G_S.

Sum Rule. If $S = A \cup B$ and $A \cap B = \emptyset$, then

$$G_s(x) = G_A(x) + G_B(x).$$

Product Rule. If $S = A \times B$ and if, for all $\sigma = (a, b)$, the weight $w(\sigma) = w(a) + w(b)$, then

$$G_s(x) = G_A(x) \cdot G_B(x).$$

Example 7. Let S be the set of all k-tuples $(a_1, a_2, \ldots, a_k)$ where the a_j's are positive integers and the weight of the element $\sigma = (a_1, a_2, \ldots, a_n)$ in S is $\sum_{j=1}^{k} a_j$. Find the corresponding generating function G_S. What does the coefficient $[x^n]G_S(x)$ count? Use the binomial theorem

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \binom{n+a-1}{n} x^n$$

to derive the formula

$$[x^n]G_S(x) = \binom{n-1}{k-1}.$$

Definition. A *partition* of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Thus a partition

$$\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m)$$

satisfies $n = \lambda_1 + \cdots + \lambda_m$ where the λ_j's are positive integers and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The number of partitions of n is denoted by $p(n)$.

Theorem [Euler].

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

Theorem [Euler]. The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Additional examples.

Example 8. Show that the number of subsets of $[n]$ containing exactly one pair of consecutive integers is

$$\sum_{k=1}^{n-1} F_k F_{n-k} = \frac{2nF_{n+1} - (n+1)F_n}{5}.$$

Example 9. Find the sequence (a_n) if $a_0 = 1$ and

$$\sum_{k=0}^{n} a_k a_{n-k} = 1, \qquad n \geq 1.$$

Example 10. Prove that the number of partitions of n in which all the even parts are distinct is the same as the number of partitions of n where each part is repeated at most three times.

Example 11. Prove that the number of partitions of n into parts not divisible by d is the same as the number of partitions of n in which no part occurs d or more times.

Example 12. Find the number of permutations of $[n]$ that have no r-cycle.