Euler

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Background

Leonhard Euler was born in 1707 in Basel, Switzerland and died in 1783 in St. Petersburg, Russia. He influenced all areas of mathematics in the Eighteenth Century, consolidating all that was known in any given branch, extending these branches, and expanding the frontiers by inventing new branches. He was a genius with a phenomenal memory, amazing insight and a selfless pursuit for the truths in mathematics. In several instances, he withheld publication of his results when he knew of other younger or less well known mathematicians, who had also arrived at the sane result. He wrote textbooks that are still models of our textbooks in use today. He did not try to hide his methods, but rather shared the excitement of discovery that he experienced. The great mathematician Laplace said **" Read Euler, read Euler. He is the master of us all."** Sadly, until recently one could not do this in English. Although the collected works of Euler take up over eighty volumes, none were available in English. A modern proponent of reading Euler was George Polya in his great book on how to do mathematics, *Mathematics and Plausible Reasoning* [**1**]. Springer-Verlag has reprinted a few of his most famous texts in English. They are *Elements of Algebra* [**2**] and *Introduction to Analysis of the Infinite*[**3**],[**4**]. Two recent publications from the Mathematical Association of America were invaluable in my preparations. They are *Journey Through Genius* and *Euler, The Master of Us All*, by William Dunham. [**5**],[**6**]. I recommend them to anyone who wants to know more about the great moments in mathematics. A major part of this talk is based on the material in these books. Dunham opens his book with an inscription for the great architect Christopher Wren who is buried in St. Paul's Cathedral. "Lector, si monumentum requiris, circumspice". Today, I say to those of you looking for the monument in honor of Euler. **" Auditor, si monumentum requiris, in mathematica circumspice".**

Geometry

One of the oldest inequalities about triangles is that relating the radii of the circumcircle and incircle. It was proved by Euler and is contained in the following theorem and corollary. Proofs are given in *Geometry Revisited* by Coxeter and Greitzer [**7**]. It is published by the Mathematical Association of America and should be on the bookshelf of everyone interested in geometry.

Theorem [Euler 1765] *Let O and I be the circumcenter and incenter, respectively, of a triangle with circumradius R and inradius r; let d be the distance OI. Then*

$$d^2 = R^2 - 2Rr$$

Corollary *In a triangle with circumradius R and inradius r, $R \geq 2r$.*

Theorem [Euler 1767] In a triangle, let O be the circumcenter, G be the centroid, and H be the orthocenter. The points O, G and H are collinear in that order and $GH = 2GO$. The line is known as the ***Euler line***.

Number Theory

Euler completed Euclid's theorem about even perfect numbers showing that there are no others than the ones Euclid found.

Theorem [Euclid Book IX.36] If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

Euler used the continuous to find insights into the discrete. He found a fantastic relation between an infinite series of reciprocals of powers of integers and an infinite product of expressions indexed by primes. This led Riemann to the most famous unsolved problem in mathematics, the Riemann Hypothesis. See the wonderful account of all of this in the recently published *Prime Obsession* [**10**].

Theorem [Euler 1737] $\sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_p \frac{1}{1-\frac{1}{p^s}}$, where p is prime.

Fermat went to his grave thinking that his conjecture that $2^{2^n}+1$ generated prime numbers for $n \geq 0$ was true. Euler in 1730 showed that $2^{2^5}+1 = 4,294,967,297$ is not prime. Although Fermat writes that he has proved his "little" theorem, the proof was never published and Euler remedied this by publishing a proof in 1731. During his lifetime he came up with at least three different proofs, including a generalization of the theorem using a function he invented, the **Euler ϕ function,** although it was Gauss who first used the Greek letter ϕ to denote the function in his great masterpiece, *Disquisitiones Arithmeticae*. The function $\phi(n)$ gives the number of positive integers less than or equal to n that are relatively prime to it.

Theorem [Fermat 1640] If p is a prime then p divides $a^p - a$.

Theorem [Euler 1750] $a^{\phi(n)} \equiv 1 \bmod n$.

Fermat also claimed to have a proof that every prime of the form $4k+1$ has a unique representation as a sum of two squares. Euler in 1749, published a proof. He was also able to show in 1751 that every number could be represented as the sum of four rational squares, but was unable to make the next step to integer squares. His young protege, Joseph Louis Lagrange, building on Euler's work, completed the proof in 1772. This must have spurred Euler on, since later in the year he published a paper where he "...begins by congratulating Lagrange on his achievement, then, rightly describing Lagrange's proof as 'far-fetched and laborious', he proceeds to give a new and elegant variant of the proof for the sum of two squares showing finally that applies equally well ... to $X^2+Y^2+Z^2+T^2$." [**11**]

Infinite Series

One of the highlights of the year in calculus classes that I teach (at least for me) , comes after the Advanced Placement exam in May. We see how the 28 year old Euler in 1735 solved a problem known as the Basel Problem. This problem had been proposed by Jakob Bernoulli in 1689 when he collected all of the work on infinite series of the 17^{th} century in a volume entitled *Tractatus De Seriebus Infinitis*. Bernoulli's comment in this volume was that the evaluation "is more difficult than one would expect". Little did he know how difficult it really was and that it would take almost 50 years for the problem to be solved. The problem is to find the sum of the reciprocals of the squares of all of the positive integers. It was known that the sum was less than two. After seeing Euler's brilliant insight and proof, we then see how the problem can be solved to the satisfaction of mathematicians today, using only knowledge gained in first year calculus. First Euler found a way to sum the series to six decimal places. (To do this term by term would require over 1,000,000 terms.) Of course, Euler found a clever way around this. If you know some calculus, see *How Euler Did It*, Dec 2003, by Ed Sandiver at MAA Online [**9**]. Then he found an even better way to make the series converge faster, now known as the **Euler-MacLaurin Summation Formula**, finding the sum to 20 decimal places. Euler then somehow recognized π in the answer and started experimenting with the infinite polynomial expansion for $\sin x$. Using analogous results from the theory of nth degree polynomials he was able to find the exact sum.

Theorem [Euler 1735] $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

Areas for Further Research

We have only scratched the surface. In November of 1983, the Mathematical Association of America journal, *Mathematics Magazine* devoted an entire issue to commemorate the two hundredth anniversary of the death of Euler [**8**]. A glossary at the end of the magazine collected over forty theorems, terms, and formulae that contain the name Euler. In geometry there is Euler's (nine-point) Circle. From the classification of polyhedra comes the Euler Characteristic and its generalization leading to topology. The Königsburg Bridge problem led to the Euler Circuit which opened up another new branch of mathematics, graph theory. To represent logical relations the Euler (Venn) Diagrams were born. In combinatorics we have the Eulerian Numbers and Euler's Theorem on Partitions. Euler's Theorem for pentagonal numbers and its connection to partitions and the pattern in the sequence of numbers generated by the sum of the divisors of n is astounding. See [**1**] pp 90-101. In attempting to answer to the question about what value should be assigned $\frac{1}{2}$!, the gamma function (Euler's Second Integral) was invented. Euler's formula, $e^{i\theta}=\cos\theta+i\sin\theta$ that provides the link between exponential functions and trigonometric functions is another jewel in the crown. Then there is Euler's constant, $\gamma=\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)\approx .57721566490153286$, which appears in many places in numerical analysis. Euler calculated its value to 16 decimal places. It is still not known whether γ is rational or irrational.

Some Problems

1. What is the answer to the following clue from a cryptic crossword in *The Guardian*? **Ideal anesthetic, such as 28. (7,6)** [The ordered pair (7,6) means the answer consists of a seven letter word followed by a six letter word.]

2. Find $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$.

3. Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

4. Find $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

References

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7. H.S.M. Coxeter, S.L. Greitzer. *Geometry Revisited*. Mathematical Association of America, 1967.

8. *A Tribute to Leonhard Euler*, Mathematics Magazine, Vol. 59 No. 5, November 1983.

9. Ed Sandifer, *How Euler Did It*, MAA Online, Nov. 2003, Dec. 2003, Jan. 2004, http://www.maa.org/news/columns.html

10. John Derbyshire, *Prime Obsession*, Joseph Henry Press, 2003.

11. André Weil, *Number Theory: An Approach Through History*, Birkhäuser Boston Inc., 1984.

12. Euclid, *The Bones: A Handy, Where-to-find-it Pocket Reference Companion to Euclid's Elements*, Green Lion Press, 2002.

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: tricycle222@earthlink.net