Berkeley Math Circle **Oct. 3, 2006** **Mathematical Games**

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Problems and exercises about Parity

1 (Colorado Mathematical Olympiad 1987) If 127 people play in a singles tennis tournament, prove that at the end of tournament, the number of people who have played an odd number of games is even.

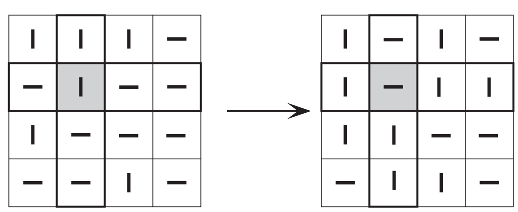
2 (Hungary 1906) Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \ldots, n$. Prove that, if n is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3) \cdots (a_n - n)$$

is an even number.

3 Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the "leapee" is at the midpoint of the line segment whose endpoints are the starting and ending position of the "leaper." Will a frog ever occupy the vertex of the square that was originally unoccupied?

4 (Bay Area Mathematical Olympiad 1999) A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see diagram). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)



before switching shaded key *after switching shaded key*

Problems and exercises about Games

For each of these, two players alternate turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works.

5 A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).

6 Can you generalize the previous game to other values (besides 16 and 4)?

7 Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. What about the general case (the starting bar is $m \times n$)?

8 Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny that is already on the table. The table starts out completely bare.

9 We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.

10 You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

11 (Bay Area Mathematical Olympiad 2002) A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. When a player makes a move that causes all the stacks to be of height 1 or 2 at the end of his or her turn, that player wins. Which starting values of n are wins for each player?

12 Start with several piles of beans. A legal move consists of removing one or more beans from a pile.

(a) Verify that this game is *very* easy to play if you start with just one pile, for example, of 17 beans.

(b) Likewise, if the game starts with two piles, the game is quite easy to analyze. Do it!

(c) But what if we start with three or more piles? For example, how do we play the game if it starts with three piles of 17, 11, and 8 beans, respectively? What about four piles? More?

13 (Putnam 1995 B5) A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking **either**

a. one bean from a heap, provided at least two beans are left behind in that heap, **or**

b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.