Mascheroni and Steiner Constructions

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Berkeley Math Circle November 14, 2006

1 History

One of the oldest games is the game invented by the Greeks, geometric construction with a straightedge and a compass. With these two simple tools configurations of great complexity can be constructed. As in all of mathematics, even when a solution to a construction has been found, the search is not over. Can a more elegant solution with fewer steps be found? With a given unit, it can be shown that the rational number field can be constructed. In addition, it can be shown that the irrational numbers arising from square roots can also be constructed, thereby extending the rational field to a new field with elements of the form $a + b\sqrt{n}$ where a, b and n are numbers in the rational field. Square roots can again be constructed in this new field and further extended, and this process can go on indefinitely. Since only these quadratic extensions can be made, there are numbers that cannot be constructed. This is the reason that certain constructions are *impossible*. In particular, the three problems from antiquity are impossible. Namely, doubling a cube, trisecting an angle, and squaring a circle.

2 Constructions Using Only a Compass

Amid all this impossibility it is rather shocking that all of the classical Euclidean constructions can be done without ever using a straightedge. Of course the lines are not drawn. Two points determine a line and that is all that is necessary to actually construct. This amazing fact was discovered in 1797 by the Italian geometer Lorenzo Mascheroni. (A bit of trivia. In 1928 an old book entitled *Euclides Danicus* was found in Copenhagen by Georg Mohr. It contained Mascheroni's result with a different proof and it was published in 1625.) Every construction is just a series of steps that involve the following four operations.

(A) To draw a circle with a given center and radius.

(B) To find the points of intersection of two circles.

(C) To find the points of intersection of a line and a circle.

(D) To find the point of intersection of two lines.

Therefore it is only necessary to show that each of these operations can be accomplished with only a compass. In order to do this only four classical constructions are needed.

(1M) To construct the reflection of a point through a line.

(2M) To construct a segment n times as long as a given segment for any integer n.

(3M) To construct the fourth proportional to three given segments.

(4M) To construct the midpoint of an arc with a given center.

3 Constructions Using Only a Straightedge

In 1833 the Swiss geometer Jakob Steiner proved that all of the classical Euclidean constructions can be done without ever using a compass, after a single circle with a given center is placed in the plane. In the same way, it is only necessary to show that the operations **A, B, C** and **D** can be effected. This can be done by constructing with only a straightedge the following five classical constructions.

(1S) To construct a line parallel to a given line thorough a point P.

(2S) To construct a perpendicular to a line through a point P.

(3S) To construct a point on a line a given distance in both directions from a point on the line.

(4S) To construct the fourth proportional to three given segments.

(5S) To construct the mean proportional of two given segments.

4 Methods for Mascheroni Constructions

(0) To construct a circle at A having a given radius BC with a collapsing compass.

A_{AB}, B_{BA}	D_{DC}, E_{EC}	A_{AF}
D, E	C, F	

(1M) To construct the reflection, Y, of X with respect to line AB.

P_{PX}, Q_{QX}	Y
X, Y	

(2M) To construct a segment n times as long as given segment AB, where n is a positive integer

A_{AB}, B_{BA}	C_{CA}, B_{BA}	D_{DC}, B_{BA}	N
C and some other point	A, D	C, N	

$AN = 2AB$. Continuing in this manner one can construct $3AB$, $4AB, \ldots$.

(3M) To construct ST, the fourth proportional to a, b, c.

O_a, L_c	L_t, O_b	M_t, O_b	ST
(L on a) M	S in the interior of $\angle$LOM	T not in the interior of $\angle LOM$	

If $c > 2a$ then by taking n large enough one can make $2na > c$. Now use the above method with na, nb, and c.

(4M) To construct the midpoint F of an arc AB with center O.

A_{AO}, O_{AB}	B_{BO}, O_{AB}	C_{CB}, D_{DA}	C_{OE}, arc AB
C	D	E	F

Operation **A**, to draw a circle with a given center and radius, and Operation **B**, to find the points of intersection of two circles, are two operations that can clearly be accomplished with only a compass. To accomplish Operation **C**, to find the points of intersection of a line and a circle, use **(1M)** to reflect circle O_r through line AB to circle O'_r. The intersection points of O_r and O'_r are the intersection points of line AB and O_r. If the reflection of O is O' (O lies on line AB), then choose any point P on the circle and use **(1M)** to reflect it through line AB to Q which will also lie on circle O. Use **(4M)** to find the midpoints of major and minor arcs PQ. These midpoints are the points of intersection of line AB and O_r. If the reflection of P is Q then P is one of the points on AB and O_r. To find the other point use **(2M)** to step around circle 0.

To accomplish Operation **D**, to find the point of intersection, F, of two lines AB and CD, first use **(1M)** to reflect C and D through line AB. Then complete parallelogram $CC'ED$ by drawing C'_{CD} and $D_{CC'}$. Note that $DD'E$ is a straight line. Since $\Delta DD'F \sim \Delta ED'C'$, use **(3M)** to construct the fourth proportional x to $D'E, DD'$, and $C'D'$. The intersection of D_x and D'_x is F.

5 Methods for Steiner Constructions

(1S) To construct a line parallel to a given line AB through a point P. If the midpoint, C, of AB is known, then take T on AP extended and find the intersection, S, of TC and PB. Find the intersection, Q, of AS and TB. PQ is parallel to AB. If the midpoint is not known then from any point X on AB draw XO through the center O of the fixed circle, intersecting the circle at L and M. Through any point of the circle R, other than L and M, use the previous construction to draw a line through R parallel to LM intersecting the circle at Q and AB at Y. Let R' and Q' be the endpoints of diameters. Draw $R'Q'$ intersecting AB at Y'. X is the midpoint of YY', so the previous construction can now be used to draw the parallel.

(2S) To construct a line through P perpendicular to line AB. Let Q be a point on the fixed circle so that the line through Q parallel to line AB does not pass through the center of the circle and intersects the circle at R. Let Q' be the endpoint of the diameter containing QO. The line through P parallel to $Q'R$ is perpendicular to line AB.

(3S) To mark off a distance $XY = k$ along line AB from a point P, first draw PM and XM parallel to XY and PY, respectively, to form parallelogram $XMPY$. Draw radii OH and OJ in the fixed circle parallel to AB and PM, respectively. Draw a line through M parallel to JH intersecting AB at Z. $PZ = k$.

(4S) To construct the fourth proportional to a, b, and c, use the previous construction **(3S)** to lay off the segments a, b, and c as in the classical construction and finish it by using construction **(1S)** to construct the parallel.

(5S) To construct the mean proportional $\sqrt{xy}$ of two segments of length x and y, let $t = x+y$ and by construction **(4S)** above, find the $m = \frac{dx}{t}$ and $n = \frac{dy}{t}$ where d is the diameter of the fixed circle. Since $m + n = d$, the $\sqrt{mn}$ can be found by the classical construction using construction **(3S)** to lay off m on the diameter of the fixed circle and using construction **(2S)** to construct a perpendicular at that point. Now we can again use construction **(4S)** to find $\frac{t\sqrt{mn}}{d}$ which is equal to $\sqrt{xy}$.

Since no circles other than the initial fixed circle are ever drawn, only two of the operations need to be addressed. Operation **C**, to find the intersection of a circle X_r and a line AB, can be accomplished by using construction **(2S)** to draw a perpendicular from X intersecting AB at T. Use construction **(3S)** to find points M and N a distance $\sqrt{(r - XT)(r + XT)}$ from T on line AB. The points M and N are the points of intersection of line AB and circle X_r.

Operation **B**, the intersection of two circles X_r and Y_m, $m \geq r$ where $XY = t$, begins by constructing $s = \frac{r^2}{2t} + \frac{t^2}{2t} - \frac{m^2}{2t}$. Each term can be constructed by using construction **(4S)** after $2t = t + t$ has been constructed using construction **(3S)** and all three can be added by using construction **(3S)**. If $t > m$ then mark off s on XY from X and if $t < m$ then mark off s on XY extended from X. At this point, construct a perpendicular line using construction **(2S)** and the problem is reduced to finding the points of intersection of this perpendicular and either of the circles, which is accomplished by the previous operation **C**.

6 Some Problems

1. Given a circle and its center, find four points on the circle that divide the circumference into four equal arcs using only a compass. (**Napoleon's Problem**).

2. A geometer has a "right-angler", an instrument that can only draw a straight line through two points and erect a perpendicular at a given point on a line. How can the geometer use this instrument to drop a perpendicular from a given point P to a given line ℓ? (**Leningrad Mathematical Olympiad, grade 7, 1987**).

3. Prove that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals. (This theorem is needed to prove the method of bisecting a given arc with only a compass.)

4. Construct the inverse of a given point outside a given circle of inversion, using only a compass and fewer than five additional circles.

5. Construct a segment half as long as a given segment using only a compass. Construct a segment one-third as long as a given segment using only a compass.

6. Given the points of a circle without the center, construct the center using only a compass. (Hint: Inversion in a circle might help.)

7. Construct a pentagon with a compass alone. (This can be done with circles of only three different radii.)

8. (**BAMO 2003**) Given a circle with a line through the center, O, intersecting the circle at A and B and a point P and not on the line in the exterior of the circle, construct, using only a straightedge, a line through P perpendicular to the line AB.

9. Given a line, two points not on the line, and the reflection of one of the points through the line, construct, using only a straightedge, the reflection of the other point through the line.

10. A parallel ruler is a straightedge with two parallel edges. Show how to bisect an angle, bisect a line segment, and construct parallel and perpendiculars to a line with a parallel ruler. (It is possible to construct, with a parallel ruler alone, all of the constructions that can be drawn with a compass and a straightedge.)

7 References

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5. George Martin, *Geometric Constructions*, Springer,1998.

6. A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, Blaisdell Publishing, 1961.

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