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Vectors - Applications to Problem Solving

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1. Well-known Facts

(1) Let A_1 and B_1 be the midpoints of the sides BC and AC of $\triangle ABC$. Prove that

$$\text{(a)}\quad \overrightarrow{AA_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}); \qquad \text{(b)}\quad \overrightarrow{B_1A_1} = \frac{1}{2}\overrightarrow{AB}.$$

(2) Let A_1 and B_1 be the midpoints of the sides BC and AD of quadrilateral $ABCD$. Prove that

$$\overrightarrow{B_1A_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}).$$

Note that Exercise 2 generalizes Exercise 1: for part (a) let D coincide with A; for part (b) let D coincide with C.

(3) Consider vector $\overrightarrow{XY}$, and draw two "paths" of vectors $\vec{v_1}, ..., \vec{v_n}$ and $\vec{w_1}, ..., \vec{w_m}$ such that each path starts at point X and ends at point Y. Prove that

$$\overrightarrow{XY} = \frac{1}{2}(\vec{v_1} + \cdots + \vec{v_n} + \vec{w_1} + \cdots + \vec{w_m}).$$

Note that Exercise 3 further generalizes Exercise 2.

(4) Let f be any of the following transformations of the plane: a rotation, a translation, a homothety, a reflection, or a composition of the above. Let $\vec{v}$ and $\vec{w}$ be two vectors in the plane. Prove that

$$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}).$$

Definition. A *distance–preserving* transformation f of the plane is a transformation which preserves all pairwise distances, i.e. for any two points A and B is the plane we have that the distance between A and B is the same as the distance between their images $f(A)$ and $f(B)$ under the transformation: $|AB| = |f(A)f(B)|$.

(5) Check that rotations, translations and reflections are distance–preserving transformations, but homotheties are *not* except in the cases of the identity transformation and central symmetries (both of which are special cases of homotheties.) Further, prove that any distance–preserving transformation of the plane is a composition of a translation and a rotation, or of a translation and a reflection.

2. Centroid and Leibnitz Theorem

Definition. Given points $A_1, A_2, ..., A_n$ (in the plane or in space), the *centroid* G of these points is the unique point which satisfies

$$\overrightarrow{GA_1} + \overrightarrow{GA_2} + \cdots + \overrightarrow{GA_n} = \vec{0}.$$

Note that the *medicenter* of any $\triangle ABC$ is the centroid of the vertices A, B, C.

(6) Prove that for any points $A_1, A_2, ..., A_n$ there exists a unique centroid G as defined above.

(7) Let G be the centroid of $A_1, A_2, ..., A_n$, and X - an arbitrary point. Prove that

$$\overrightarrow{XA_1} + \overrightarrow{XA_2} + \cdots + \overrightarrow{XA_n} = n\,\overrightarrow{XG}.$$

Note that for $X = G$, this reduces to the definition of the centroid G.

(8) (Leibnitz) Let G be the medicenter of $\triangle ABC$, X - an arbitrary point. Prove that

$$XA^2 + XB^2 + XC^2 = 3XG^2 + GA^2 + GB^2 + GC^2.$$

Generalize to an arbitrary polygon $A_1A_2...A_n$ with centroid G:

$$\sum_{i=1}^{n} XA_n^2 = nXG^2 + \sum_{i=1}^{n} GA_i^2.$$

(9) Let H be the orthocenter of $\triangle ABC$ and let R be its circumradius. Prove that

$$HA^2 + HB^2 + HC^2 \geq 3R^2.$$

When is equality obtained?

(10) (84.49) Let $\triangle ABC$ with medicenter G be inscribed in a circle of center O. Point M lies inside the circle with diameter OG. Lines AM, BM and CM intersect the circumcircle again in points A', B' and C', respectively. Prove that the area of $\triangle ABC$ is not greater than the area of $\triangle A'B'C'$.

(11) (G260) A point M and a circle k are given in the plane. If $ABCD$ is an arbitrary square inscribed in k, prove that the sum $MA^4 + MB^4 + MC^4 + MD^4$ is independent of the positioning of the square. Replace now the square by a regular n-gon $A_1A_2...A_n$. Let $S_m = \sum_i MA_i^m$. For what natural m is S_m independent of the position of the n-gon (still inscribed in k)?

(12) (G270) Points $A_1, A_2, ..., A_n$ ($n \geq 3$) lie on a circle with center O. Drop a perpendicular through the centroid of every $n-2$ of these points towards the line determined by the remaining two points. Prove that the $\binom{n}{2}$ thus drawn lines are all concurrent.

(13) (G271) Points $A_1, A_2, ..., A_n$ ($n \geq 2$) lie on a sphere. Drop a perpendicular through the centroid of every $n-1$ of these points towards the plane, tangent to the sphere at the remaining n-th point. Prove that the n drawn lines are all concurrent.

3. Rotations and Similarities

(14) Let $\rho(O,\alpha)$ be a rotation about angle α and centered at point O. Let g be a line in the plane, and g' be its image under the rotation. Let M and M' be the feet of the perpendiculars dropped from O to g and g', and let M_1 be the intersection point of g and g'. Prove that

- (a) the angle between g and g' equals α.
- (b) one can map point M into M_1 by composing a rotation $\rho_1(O,\alpha/2)$ and a homothety $h(O,1/(\cos\frac{\alpha}{2}))$, i.e. a *similarity* $s(O,\alpha,1/(\cos\frac{\alpha}{2}))$.

(15) (G262) $\triangle ABC$ is rotated to $\triangle A'B'C'$ around its circumcenter O by angle α. Let A_1, B_1 and C_1 be the intersection points of lines BC and $B'C'$, CA and $C'A'$, and AB and $A'B'$, respectively. Prove that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar, and find the ratio of their sides.

(16) (G263) The quadrilateral $ABCD$ is inscribed in a circle k with center O, and the quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB, $B'C'$ and BC, $C'D'$ and CD, and $D'A'$ and DA. Prove that $A_1B_1C_1D_1$ is a parallelogram.

(17) (G264) In quadrilateral $ABCD$, the diagonals intersect in point O. Quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB, $B'C'$ and BC, $C'D'$ and CD, and $D'A'$ and DA. Prove that $A_1B_1C_1D_1$ is cyclic if and only if $AC \perp BD$.

(18) Let $A_1A_2A_3A_4$ be an arbitrary cyclic quadrilateral. Denote by H_1, H_2, H_3 and H_4 the orthocenters of $\triangle A_2A_3A_4$, $\triangle A_3A_4A_1$, $\triangle A_4A_1A_2$ and $\triangle A_1A_2A_3$, respectively. Prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

(19) (Kazanluk'97 X) Point F on the base AB of trapezoid $ABCD$ is such that $DF = CF$. Let E be the intersection point of the diagonals AC and BD, and O_1 and O_2 be the circumcenters of $\triangle ADF$ and $\triangle BCF$, respectively. Prove that the lines FE and O_1O_2 are perpendicular.

(20) (Bulgaria'00) Point D is a midpoint of the base AB of the acute isosceles $\triangle ABC$. Let $E \neq D$ be an arbitrary point on the base, and O - the circumcenter of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC intersect in one point.

4. Compositions of Rotations

(21) Prove that the composition of two rotations $\rho_1(O_1,\alpha_1)$ and $\rho_2(O_2,\alpha_2)$ about different centers O_1 and O_2 is:

- (a) rotation if $\alpha_1+\alpha_2 \neq k\pi$ $(k \in \mathbb{Z})$;
- (b) translation if $\alpha_1+\alpha_2 = 2k\pi$ $(k \in \mathbb{Z})$;
- (c) central symmetry if $\alpha_1+\alpha_2 = (2k+1)\pi$ $(k \in \mathbb{Z})$.

(22) (G267) On the sides of a convex quadrilateral draw externally squares. Prove that the quadrilateral with vertices the centers of the squares has equal perpendicular diagonals.

(23) (G268) Given two equally oriented equilateral triangles AB_1C_1 and AB_2C_2 with centers O_1 and O_2, respectively, let M be the midpoint of B_1C_2. Prove that $\triangle O_1MB_2 \sim \triangle O_2MC_1$.

(24) (G269) A hexagon $ABCDEF$ is inscribed in a circle of radius r so that $AB = CD = EF = r$. Let the midpoints of BC, DE, FA be L, M, N respectively. Prove that $\triangle LMN$ is equilateral.

(25) (Napoleon) If three equilateral triangles ABC_1, BCA_1 and CAB_1 are constructed off the sides of $\triangle ABC$, show that the centers of these equilateral triangle form another equilateral triangle. Prove also that AA_1, BB_1 and CC_1 are concurrent and have same lengths. Can you identify the medicenter of $\triangle O_1O_2O_3$ with some distinguished point of $\triangle ABC$?

5. Metric Relations and Geometric Loci of Points

(26) (Stuard) Prove that if point D lies on the side BC of $\triangle ABC$, and $BC = a$, $CA = b$, $AB = c$, $BD = m$, $CD = n$, $AD = d$, then $d^2a = b^2m + c^2n - amn$. In particular, for the median AM in $\triangle ABC$ we have

$$4AM^2 = 2(b^2 + c^2) - a^2.$$

(27) (Kazanluk'95 X) Given $\triangle ABC$ with sides $AB = 22,\ BC = 19,\ CA = 13$,
   (a) If M is the medicenter of $\triangle ABC$, prove that $AM^2 + CM^2 = BM^2$.
   (b) Find the locus of points P in the plane such that $AP^2 + CP^2 = BP^2$.
   (c) Find the minimum and maximum of BP if $AP^2 + CP^2 = BP^2$.

(28) (G272) Given $\triangle ABC$, find the locus of points M in the plane such that $MA^2 + MB^2 = MC^2$.

(29) (G273) Given tetrahedron $ABCD$, find the locus of points M in such that $MA^2 + MB^2 + MC^2 = MD^2$. How about $MA^2 + MB^2 = MC^2 + MD^2$?

(30) (UNICEF'95) Given a fixed segment AB and a constant $k > 0$, find the locus of points C in the plane such that in $\triangle ABC$ the ratio of some side to the altitude dropped to this side equals k.

(31) (UNICEF'95) We are given $\triangle ABC$ in the plane. A rectangle $MNPQ$ is called *circumscribed* around $\triangle ABC$ if on each side of the rectangle there is at least one vertex of the triangle. Find the locus of all centers O of the rectangles $MNPQ$ circumscribed around $\triangle ABC$.

(32) (84.22) The orthogonal projections of a right triangle onto the planes of two faces of a regular tetrahedron are themselves regular triangles of sides 1. Find the perimeter of the right triangle.

(33) (84.42) Given a pyramid $SABCD$ whose base is the parallelogram $ABCD$. Let N be the midpoint of BC. A plane γ moves in such a way that it always intersects lines SC, SA and AB in points P, Q and R and

$$\frac{\overline{CP}}{\overline{CS}} = \frac{\overline{SQ}}{\overline{SA}} = \frac{\overline{AR}}{\overline{AB}}.$$

Point M on line SD is such that line MN is parallel to plane γ. Find the locus of points M as γ runs over all possible positions.

HINTS

1 Use vectors. If O is the center of k, then $\overrightarrow{MA} = \overrightarrow{MO} + \overrightarrow{OA}$. The sum equals $4(MO^4 + 4MO^2R^2 + R^4)$, where R is the radius of k.

3 Let M be an arbitrary point in the plane, and g be a line perpendicular to OM. Denote by g' the image of g under rotation $\rho(O, \alpha)$, and by M_1 the intersection of g and g'. Consider the composition $\phi = \theta\rho_1$ where $\rho_1(O, \alpha/2)$ is rotation, and $\theta(O, k = 1/(\cos(\alpha/2)))$ is a homothety. Then ϕ is a homothety with ratio $k = 1/(\cos(\alpha/2))$. In our case, $\triangle ABC \sim \triangle A_0B_0C_0$ with ratio $k = 1/2$ (A_0 is the midpoint of BC, etc.), and $\triangle A_0B_0C_0 \sim \triangle A_1B_1C_1$ with ratio $k = 1/(\cos(\alpha/2))$. Hence the ratio of similarity between $\triangle ABC$ and $\triangle A_1B_1C_1$ is $k = 1/2(\cos(\alpha/2))$ (If $\alpha = \pi$, then $BC||B'C'$ and $\triangle_1B_1C_1$ doesn't exist.

4-5 Use Problem 4. In particular, $kA_1B_1 = AC = kC_1D_1$.

6 Let OQ be a perpendicular to g with $Q \in g$, and the line OQ intersect k in P and S with P between S and Q. Let M and N be the points of tangency of k and k_1, and k_1 and g, respectively. Show that $SP \cdot SQ = SM \cdot SN$. Repeating for k_2, conclude that S has the same tangential distance from k_1 and from k_2, hence ST is tangent to k_1 and k_2. The locus of T is a circle with center S and radius $\sqrt{SP \cdot SQ}$, minus the two diametrically opposite points on a line through S parallel to g.

7 Apply inversion fixing k and sending K to a line. Reduce to Problem 6.

8 Express the two diagonals as vector sums of all vectors pointing from a vertex of the original quadrilateral to a corresponding center of a square; then use rotation by 90° argument.

9 Consider the rotations $\rho_1(O_1, 2\pi/3)$ and $\rho_2(B_2, \pi/3)$.

10 Show that $\overrightarrow{LN}$ maps to $\overrightarrow{LM}$ after rotation by 60° around L.

12 Let G be the centroid of all points, and G' the centroid of $A_1, ..., A_{n-2}$. The line through G' perpendicular to $A_{n-1}A_n$ is parallel to $\overrightarrow{OA}_{n-1} + \overrightarrow{OA}_n$. Correspondingly, for every point S on this line we have:

$$\overrightarrow{OS} = \overrightarrow{OG'} + \lambda(\overrightarrow{OA}_{n-1} + \overrightarrow{OA}_n) = \frac{n}{n-2}\overrightarrow{OG} + (\lambda - \frac{1}{n-2})(\overrightarrow{OA}_{n-1} + \overrightarrow{OA}_n).$$

For $\lambda = n/(n-2)$ we have $\overrightarrow{OS} = \frac{n}{n-2}\overrightarrow{OG}$.

13 Construct parallelogram $AGBC$ with diagonal AB and GC. If O and N are the midpoints of AB and MC, respectively, then $MG^2 = 4NO^2 = CA^2 + CB^2 - AB^2$ - use the formula for the median $4m_c^2 = 2a^2 + 2b^2 - c^2$.