Problems and solutions from the advanced BAMO practice session
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The problems:

1. Prove that given any five points in the plane, no three collinear, some four of them form the vertices of a convex quadrilateral.

2. Let $r \geq 1$ be a real number with the property that for any positive integers m, n such that m divides n, $\lfloor mr \rfloor$ divides $\lfloor nr \rfloor$. (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) Prove that r is an integer.

3. Let ABC be a triangle with a right angle at A, and let H be the foot of the altitude from A. Prove that the radii of the inscribed circles of ABC, ABH, ACH add up to the length of the segment AH.

4. Find all triples (x, y, z) of positive integers such that $3^x + 4^y = 5^z$.

5. Let a, b, c be positive real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

The solutions:

1. We first note that given any four points in the plane, no three collinear, either they form a convex quadrilateral or one of them lies inside the triangle formed by the other three. The easiest way to see this is to draw three of the points and the three lines joining them, then notice what happens if the fourth point lands inside each of the seven regions formed by the three lines. (Say A, B, C are three of the points. If the fourth point D lands inside the triangle ABC, we are done. If it lies within the angle formed by the lines AB and AC away from the triangle, then BCD contains A, and so on.)

   Take four of the points. If they form a convex quadrilateral, we are done, so suppose they do not. Then one of the points, say D, lies within the triangle formed by the other three, say A, B, C. The fifth point E must lie within one of the angles $\angle ADB, \angle BDC, \angle CDA$. Without loss of generality, say it lies within $\angle ADB$. Then $ADBE$ is a convex quadrilateral.

2. I use $\{x\}$ to mean the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$. Suppose r is not an integer. Pick an integer m such that $mr \geq 2$ and mr is not an integer. (This is easy if r is irrational. If r is rational, just make sure m is large enough and not a multiple of the denominator of r.) Then $0 < \{mr\} < 1$; let t be the smallest integer such that $t\{mr\} \geq 1$. Then for $i = 1, \dots, t-1$, we have $\{imr\} = i\{mr\}$, so $\lfloor imr \rfloor = i\lfloor mr \rfloor$. But $\{tmr\} = t\{mr\} - 1$, so $\lfloor tmr \rfloor = t\lfloor mr \rfloor + 1$, which is not a multiple of $\lfloor mr \rfloor$ because the latter is at least 2.

   Note: various solutions are possible. One that I gave at the talk uses the following lemma whose proof I leave to you: for any real numbers x and y, $\lfloor x + y \rfloor$ is always equal to either $\lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

3. We first recall a general fact about the inscribed circle of a triangle. Let ABC be an arbitrary triangle, and suppose the inscribed circle of the triangle touches BC at X, CA at Y, and AB at Z. Then

   $$AY = AZ = \frac{AB + CA - BC}{2}$$

   and so on. This comes from the equalities $AY = AZ, BZ = BX, CX = CY$ from equal tangents (two tangents from the same point to the same circle have the same length) and the equalities $BX + XC = BC, CY + YA = CA, AZ + ZB = AB$ by solving the resulting system of linear equations.

   In the case at hand, A is a right angle. If we let I denote the center of the inscribed circle and r the inradius, then $AYIZ$ is a rectangle and $IY = IZ = r$, so $AYIZ$ is actually a square. Hence $r = AY = (AB + AC - BC)/2$.

   Applying the above formula to ABH and ACH yields that their inradii are $(AH + BH - AB)/2$ and $(AH + CH - AC)/2$. Adding these three up, we can cancel BH with $AH + CH$, AB with AB and AC with AC to get a sum of $(AH + AH)/2 = AH$.

Alternate solution: in any triangle, the area is equal to half the inradius times the perimeter. (Hint: draw the triangles AIB, BIC, CIA, where I is again the center of the inscribed circle.) Thus the inradius of ABC is equal to $(BC \times AH)/(AB + BC + CA)$. As for the other triangles, note that ABH is similar to ABC with similarity ratio AB/BC, and likewise ACH is similar to ABC with similarity ratio AC/CH, and so on.

4. Clearly $(x, y, z) = (2, 2, 2)$ is a solution, and we will show that there are no others. We first work modulo some small numbers. Taking the equation $3^x + 4^y = 5^z$ modulo 3, we get $1 \equiv 2^z \pmod 3$, which only happens when z is even. So we may write $z = 2a$ for some positive integer a. Next, working modulo 4, we see that $3^x \equiv 1 \pmod 4$, which only happens when x is even, so we may write $x = 2b$ for some positive integer b.

   Now rewrite the given equation as

$$3^{2b} = 5^{2c} - 2^{2y} = (5^c - 2^y)(5^c + 2^y).$$

   This means both $5^c - 2^y$ and $5^c + 2^y$ are powers of 3. If $5^c - 2^y > 1$, then $5^c + 2^y > 1$ also and both $5^c - 2^y$ and $5^c + 2^y$ must both be multiples of 3; but their difference is 2^{y+1}, which is not divisible by 3, contradiction. Thus we must have $5^c - 2^y = 1$ and $5^c + 2^y = 3^{2b}$. Eliminating c yields $2^{y+1} + 1 = 3^{2b}$.

   Now repeat the argument with the new equation

$$2^{y+1} = 3^{2b} - 1 = (3^b - 1)(3^b + 1);$$

   again, both $3^b - 1$ and $3^b + 1$ are powers of 2, but their difference is only 2. That can only happen if $3^b - 1 = 2$ and $3^b + 1 = 4$ (we can't have $3^b - 1 = 1$ because then $(3^b + 1) - (3^b - 1)$ would be odd, and we can't have $3^b - 1 \geq 4$ or else $(3^b + 1) - (3^b - 1)$ would be a multiple of 4). Now we can unwind everything: we have $b = 1$ and so $x = 2$; we have $2^{y+1} + 1 = 3^{2b}$ and so $y = 2$; and we have $5^c - 2^y = 1$ and so $c = 1$ and $x = 2$.

5. First suppose one of a, b, c is at most 2; without loss of generality, say $c \leq 2$. Then

$$\begin{aligned} a^2 + b^2 + c^2 &\geq a^2 + b^2 \\ &\geq 2ab && \text{[arithmetic-geometric mean inequality]} \\ &\geq abc && \text{[since } c \leq 2\text{]}, \end{aligned}$$

   so we are done in that case.

   On the other hand, if $a, b, c \leq 2$, then $a^2 \geq a$, $b^2 \geq b$, $c^2 \geq c$, so

$$a^2 + b^2 + c^2 \geq a + b + c \geq abc$$

   by assumption. So we are done in all cases.