Perennnial Problems from Geometry

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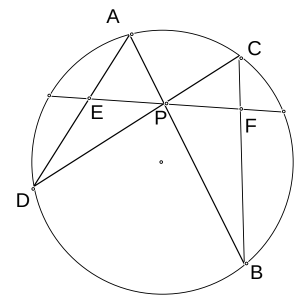


1 History and Background

Problem. Through the midpoint P of a chord of a circle, any two chords, AB and CD, are drawn. If AD and BC intersect the first chord at E and F, prove that $PE = PF$.

This geometry problem which is the major focus of the talk today dates back to at least 1815, when it appeared as question 1029 in *The Gentleman's Diary*. One of the solvers was W.G. Horner whose name is associated with Horner's Method of approximating zeros of polynomials. In R.A. Johnson's treatise of 1929 [**11**], a proof by Mackay from the 1884 proceedings of the Edinburg Math Society is given to the more general theorem:

Theorem. Given a complete quadrangle inscribed in a circle, if any line cuts two opposite sides at equal distances from the center of the circle, it cuts each pair at equal distances from the center.

It surfaced several times in the 20's and 30's in School Science and Mathematics. Then in 1943, it showed up in the elementary problem section of the American Mathematical Monthly as E 571 [**13**]. In 1965, M.S. Klamkin conjectured that the property could be extended to ellipses and in 1969, with Chakerian and Sallee [**4**], proved it. Eve's in the revised edition of *A Survey of Geometry* [8], gave a proof using poles and harmonic division from projective geometry. He stated that the problem "is a real stickler if one is limited to the use of only high school geometry". Later in the book he extended the theorem to all proper conics.

I first encountered the problem in the late 1960's taking the required "Mathematics for Education" at San Francisco State College while I was working to get my credential. I had avoided all geometry classes while getting my degree and only had the geometry I learned in the 50's in high school to attack the problem. I was unable to solve it without a hint. The hint was to construct a line through F parallel to AD and extend CD to meet this line. My proof was written in two column format and had 42 steps, not including the constructions. I thought that geometry proofs were actually written this way. It wasn't until much later that I realized that the two column format is only "training wheels" for beginners and those who have trouble writing coherent sentences. The way a mathematician would write this up can be seen in [**13**] In preparing for this talk, I looked for my proof and found it with a note attached from a student in a Junior High School geometry class I taught. I had loaned it to him as a possible solution to the problem. When he returned it over six years later he was at U.C. Berkeley on his way to becoming an optometrist. During this period when

I was teaching Junior High School I came across a lovely solution in the Twenty-Eighth Yearbook of the National Council of Teachers of Mathematics, *Enrichment Mathematics for High School*[14]. I was then a member of the Mathematical Association of America and received *Mathematics Magazine*. In 1973, Steven Conrad of California Mathematics League fame, published a proof in *Mathematics Magazine* [**5**] that did not require any auxiliary lines. In 1983, Ross Honsberger, who was just beginning to surface as a great expositor on problem solving, had a department in the *Two-Year College Mathematics Journal* (Now just called *College Mathematics Journal*) which went under the heading of "Mathematical Gems". In the January issue [**10**], he subtitled the section with *The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry—Part 1.* He gives a proof similar to the NCTM proof, but then introduces a lemma by a colleague at the University of Waterloo, Hiroshi Haruki. The butterfly problem is just a couple of steps of algebra after that. In Coxeter and Greitzer[**5**], there is a reference to the shortest proof in Coxeter's *Projective Geometry, Blaisdell, 1964* which I do not have and there is also a proof that is "simple and easy to remember". In 1976 we got *A Double Butterfly Theorem* [**12**] and in 1990, using the lemma of Haruki, *A New Proof of the Double Butterfly Theorem*[**9**]. Also in 1976, an article appeared in *Crux Mathematicorum* [**16**] that gives an account similar to mine with a few more references. While searching the internet during the winter break I came across a wonderful geometry section with the Butterfly Theorem and the Double Butterfly Theorem at Alexander Bogomolny's Cut-the-Knot website[**2**]. Here you will find the Coxeter and Greitzer proof, two by Shklyarsky from the second volume of *Selected Problems and Theorems of Elementary Mathematics*, Moscow, 1952 which was never translated into English, and proofs of the Butterfly Theorem and the Double Butterfly Theorem by Prasolov [**15**]. You will also be able to download a pdf file [**7**] of a handwritten proof by Edsger Dÿkstra that he wrote in 1983. As you can see, the problem has a long and varied history which I wanted to pass on to you. As I was finishing up this paper I came across an article that I missed in my preparations. It is an updated version(1987) of the history of the Butterfly Problem [**1**] by Leon Bankoff, with some solutions that I had never seen and listing forty-seven references.

2 Some Solutions

1. (AMM 1943) Let's start with the first one I did. Remember the hint: construct a line through F parallel to AD and extend CD to meet this line at G. Use the intersecting chords theorem to find DE in terms of the original chord, PE and AE. But $\Delta PED \sim \Delta PFG$, so find GF in terms of PF and DE. Let line GF intersect AB at H. Find FH in terms of AE and note that $GCHB$ is cyclic and BC is a common chord of this new circle and the original circle.

2. (NCTM Yearbook1963) Reflect D through the perpendicular bisector of the original chord to D'. Connect $D'F$, $D'P$ and $D'B$. Since $\angle FPD' = \angle D'BF$, $BPFD'$ is cyclic and $\Delta DPE \cong \Delta D'PF$.

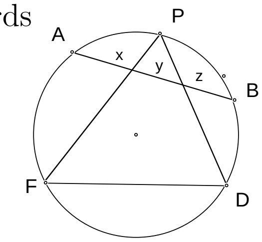
3. (Conrad 1973) Let $[ABC]$ denote the area of ΔABC. Note the following:

$$\frac{[PEA][PFC][PED][PFB]}{[PFC][PED][PFB][PEA]} = 1$$

   Now use the area formula $\frac{1}{2}ab\sin C$ to find each of the areas and simplify.

4. (Coxeter and Greitzer) Drop perpendicular segments from E to PA and PD and drop perpendicular segments from F to PB and PC. Using similar right triangles, write down the proportionalities relating the perpendicular segments to PE, PF, AE, CF, DE, and BF. Multiply them all together and simplilfy.

5. (Hiroshi Haruki's Lemma) Suppose AB and FD are nonintersecting chords in a circle and that P is a variable point on the arc AB, remote from F and D. Then, for each position of P, the lines PF and PD cut AB into three segments of lengths x, y, z such that $\frac{xz}{y}$ = a constant k.



6. (Mathematical Gems 1983) Apply Haruki's Lemma twice and simplify.

7. (The Double Butterfly) Let PQ be a fixed chord of a cicle and let "butterfly R" and "butterfly S" be inscribe in the circle and oriented such that their wings cut PQ (in order from left to right) at R_4, R_3, R_2, R_1 and S_1, S_2, S_3, S_4, respectively. If $PR_1 = QS_1$, $PR_2 = QS_2$, and $PR_3 = QS_3$, then $PR_4 = QS_4$.

8. (Dÿkstra's Analytic Proof) Let the circle have radius 1 and center at the origin. Let the fixed chord be parallel to the y-axis and midpoint P be located at $(p, 0)$. Locate points A,B,C,and D, each as the intersection of two lines from $(-1, 0)$ and $(1, 0)$. Each pair can be parameterized by a single variable, $\alpha, \beta, \gamma, \delta$, respectively, since they are perpendicular. For a given point multiply one of the equations by the parameter for another point and subtract from the second equation, cleverly giving an equation symmetric in the both parameters. It is therefore an equation for the chord joining those two points. Using the coordinates of P in the equations for chords AB and CD gives $\alpha : \beta = \gamma : \delta$. Plugging p into the symmetric equation shows that the y coordinates of E and F are equal in magnitude.

3 More Problems

Here are some problems that can be solved using the area addition property in the plane.

1. Suppose that Cevians AD, BE and CF are concurrent at point P of the interior of triangle ABC. Prove that $\frac{DP}{AD} + \frac{EP}{BE} + \frac{FP}{CF} = 1$.

2. Let triangle ABC be acute and let H be its orthocenter. The altitudes AA_1, BB_1, and CC_1. Prove that $\frac{AH}{AA_1} + \frac{BH}{BB_1} + \frac{CH}{CC_1} = 2$.

3. Let triangle ABC be isosceles with $AB = AC$. The altitude from A is AE and cevian BF intersects AE at D. If $AF : AC = 1 : 3$ Then find $AD : DE$ and $BD : BF$.

4. Quadrilateral $ABCD$ is inscribed in a circle. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$ and $BD = q$. Prove Ptolemy's second theorem that $\frac{p}{q} = \frac{ad + bc}{ab + cd}$

4 References

1. Leon Bankoff, *The Metamorphosis of the Butterfly Theorem*, Mathematics Magazine, Mathematical Association of America, October 1987.

2. Alexander Bogomolny, *The Butterfly Theorem* and *The Two Butterfly Theorem*, http://www.cut-the-knot.com/geometry.shtml, number 30.

3. Jim Boyd, Richard Palmaccio, Willie Yong, *Adventures in Area Analysis*, Mathematics & Informatics Quarterly, Volume 12, Number 1, 2002.

4. G.D. Chakerian, G.T. Sallee, M.S. Klamkin,*On the Butterfly Property*, Mathematics Magazine, Mathematical Association of America , January 1969.

5. Steven R. Conrad, *Another Simple Solution of the Butterfly Problem*, Mathematics Magazine, Mathematical Association of America, November 1973.

6. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, Mathematical Association of America, 1967.

7. Edsgar Dy̌kstra, *An Analytical Proof of the Butterfly Theorem*, November 1983, PDF file can be found at Alexander Bogomolny's website noted in reference [**1**].

8. Howard Eves, *A Survey of Geometry*, Allyn and Bacon, Revised Edition 1972.

9. Larry Hoehn, *A New Proof of the Double Butterfly Theorem*, Mathematics Magazine, Mathematical Association of America, October 1990.

10. Ross Honsberger, *Mathematical Gems*, The Two-Year College Mathematics Journal, Mathematical Association of America, January 1983.

11. R. A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, 1960.

12. Dixon Jones, *A Double Butterfly Theorem*, Mathematics Magazine, Mathematical Association of America, March 1976.

13. Sol Mitchell, et.al., E 571[1943,346], American Mathematical Monthly, February 1944.

14. Charles Pinzka, *Problem Solving and Some Problems*, 28th NCTM Yearbook, 1963.

15. V.V. Prasolov, *Essays on Numbers and Figures*, American Mathematical Association, 2000.

16. Léo Sauvé, *The Celebrated Butterfly Problem*, Volume 2, Number 1, Eureka (Now Crux Mathematicorum), 1976.

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us