Loose Ends from Previous Talks

Tom Rike

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1 Problems

1. (Hiroshi Haruki's Lemma) Suppose AB and FD are nonintersecting chords in a circle and that P is a variable point on the arc AB, remote from F and D. Then, for each position of P, the lines PF and PD cut AB into three segments of lengths x, y, z such that $\frac{xz}{y}$ = a constant k.

2. (Heron's Proof) The area of a triangle is equal to $\sqrt{s(s-a)(s-b)(s-c)}$.

3. (Euler's Proof) The area of a triangle is equal to $\sqrt{s(s-a)(s-b)(s-c)}$.

   Here are some problems that can be solved using the area addition property in the plane.

4. Suppose that Cevians AD, BE and CF are concurrent at point P of the interior of triangle ABC. Prove that $\frac{DP}{AD} + \frac{EP}{BE} + \frac{FP}{CF} = 1$.

5. Let triangle ABC be acute and let H be its orthocenter. The altitudes AA_1, BB_1, and CC_1. Prove that $\frac{AH}{AA_1} + \frac{BH}{BB_1} + \frac{CH}{CC_1} = 2$.

6. Let triangle ABC be isosceles with $AB = AC$. The altitude from A is AE and cevian BF intersects AE at D. If $AF : AC = 1 : 3$ Then find $AD : DE$ and $BD : BF$.

7. Quadrilateral $ABCD$ is inscribed in a circle. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$ and $BD = q$. Prove Ptolemy's second theorem that $\frac{p}{q} = \frac{ad+bc}{ab+cd}$

   Use the idea of excenters and incenters to solve the following problems.

8. (Carleton University Mathematics Competition for High School Students, 1976) ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

9. (Pythagoras Olympiad in The Netherlands, 1980) In triangle ABC, point D is such that $\angle DCA = \angle DCB = \angle DBC = 10°$ and $\angle DBA = 20°$. Find the measure of $\angle CAD$.

10. (Alberta High School Mathematics Competition, 1989–90) In quadrilateral $ABCD$ with diagonals BD and AC, $\angle ABD = 40°$, $\angle CBD = 70°$, $\angle CDB = 50°$, $\angle ADB = 80°$. Find the measure of $\angle CAD$.

11. (Junior Problem A-6, Tournament of Towns, Spring 1997) Let P be a point inside triangle ABC with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$ and $\angle ACP = 30°$. Find the measure of $\angle BPC$.

12. Senior Problem A-2, Tournament of Towns, Spring 1997) D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find the measure of $\angle A$.

13. In $\triangle ABC$, D, E, and F are the trisection points of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ nearer A,B,C, respectively. Let $\overline{BF} \cap \overline{AE} = J$. Show that $BJ : JF = 3 : 4$ and $AJ : JE = 6 : 1$.

14. In the previous problem, let $\overline{CD} \cap \overline{AE} = K$ and $\overline{CD} \cap \overline{BF} = L$. Use the previous problem to show that $DK : KL : LC = 1 : 3 : 3 = EJ : JK : KA = FL : LJ : JB$.

15. Use the previous two problems to show that the triangle $\triangle JKL$ is one-seventh the area of $\triangle ABC$. Generalize the problem using points which divide the sides in a ratio of $1 : n$ to show the ratio of the areas is $(1-n)^3 : (1-n^3)$. This can be generalized even further using different ratios on each side. It is known as Routh's Theorem. See [**2**] [**5**] and [**8**].

16. (**AIME 1985 #6**) In triangle ABC, cevians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect at point P. The areas of triangles PAF, PFB, PBD and PCE are 40,30,35 and 84, respectively. Find the area of triangle ABC.

17. (**AIME 1988 #12**) Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$ and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$ then find abc.

18. (**AIME 1989 #15**) Point P is inside triangle ABC. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.

19. (**AIME 1992 #14**) In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find the value of $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

20. (Larson [**14**] problem 8.3.4) In triangle ABC, let D and E be the trisection points of BC with D between B and E. Let F be the midpoint of $\overline{AC}$, and let G be the midpoint of $\overline{AB}$. Let H be the intersection of $\overline{EG}$ and $\overline{DF}$. Find the ratio $EH : HG$.

21. (Mandelbrot March 2003) The square of the area of a quadrilateral that admits an inscribed circle is equal to $(a + b + c + d)(abc + abd + acd + bcd)$ where a, b, c, and d are the lengths of the tangent segments from the vertices to the incircle.

22. (Theorem) The points A_1, B_1, C_1 are chosen on the sides of triangle ABC (A_1 on $BC, etc.$). The segments AA_1, BB_1 and CC_1 intersect at one point if and only if $\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin ACC_1}{\sin BCC_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} = 1$

2 Some Hints and Answers

16. 315

17. 441 (Show $\frac{d}{a+d} + \frac{d}{b+d} + \frac{d}{c+d} = 1$.)

18. 108 (Show CP:PF = 3:1. Draw a line segment from D to the midpoint of PB. Notice that it forms a 3-4-5 triangle which is one-eighth of the total area.

19. 94 (Assign weights of x, y, z to the vertices, find the ratios and multiply.)

20. 2 : 3 (First draw $\overline{GC}$ intersecting $\overline{DF}$ at K. Find $CK : KG$. Now work on triangle DCG.)

3 References

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If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us