BERKELEY MATH CIRCLE 2000-2001

Practice Exam II for BAMO 2001

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1. At a round table there are (a) 15; (b) 20 people. They want to reseat themselves in such a way that those who were sitting before next to each other have now two poeple between them. Is this possible?

2. A student is swimming in the center of a square pool. Suddenly, a teacher comes up to one of the corners of the pool. The teacher cannot swim, but he can walk 4 times faster than the student swims. The student runs faster than the teacher. Can the student run away?

3. Prove that if the integers a_1, a_2, ..., a_n are all distinct, then the polynomial
$$(x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$$
cannot be written as a product of two other polynomials with integer coefficients.

4. Let $\triangle ABC$ be an acute triangle. Prove that there exist unique points A_1, B_1 and C_1 lying respectively on sides BC, AC and AB, with the property: Each of the three points is the midpoint of the segment whose ends are the orthogonal prjections of the other two points onto the corresponding side. Further, prove that $\triangle A_1B_1C_1$ is similar to the triangle which can be formed by the medians of $\triangle ABC$.

5. Let A be the set of all sequences of length n consisting of 0's and 1's. Denote by O the sequence in A consisting only of 0's. The sum of two sequences $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ...b_n)$ is a third sequence $c = (c_1, c_2, ..., c_n)$ for which $c_i = 0$ if $a_i = b_i$ and $c_i = 1$ if $a_i \neq b_i$. (For example, (0,0,1)+ (0,1,1)=(0,1,0).) Let $f : A \to A$ be the function satisfying the following conditions: $f(O) = O$; if sequences a and b differ in exactly k places, then sequences $f(a)$ and $f(b)$ also differ in exactly k places. Prove that for if a, b, c are sequences in A for which $a+b+c = O$, then $f(a)+f(b)+f(c) = O$.

6. **(For the die-hards. Don't try this at home alone!)** Let $p \geq 3$ be a prime number, and let $a_1, a_2, ..., a_{p-2}$ be a sequence of natural numbers such that p does **not** divide a_k and $a_k^k - 1$ for all $k = 1, 2, ..., p-2$. Prove that we can choose some elements of the sequence such that their product has remainder 2 when divided by p.