Berkeley Math Circle: Monthly Contest 6 Solutions

1. A fair coin is flipped nine times. Which is more likely, having exactly four heads or having exactly five heads?

Solution. Both outcomes are equally likely! Notice that for every sequence with four heads (such as \textit{HTTHHTHTT}) there is a corresponding sequence with exactly five heads formed by reversing all the coin flips (in this case \textit{THHTHTHH}). Thus the number of sequences of nine flips with exactly four heads is equal to the number of sequences with exactly five heads; since every sequence is equally likely this completes the proof.

In fact, both probabilities are equal to \( \binom{9}{4} = \binom{9}{5} = \frac{126}{512}. \)

2. Let \( a \) and \( b \) be positive real numbers. Prove that

\[
\sqrt{a^2 - ab + b^2} \geq \frac{a + b}{2}.
\]

Solution. Squaring both sides (which is OK since both sides are positive), it’s equivalent to show that \( 4(a^2 - ab + b^2) \geq (a + b)^2. \) But their difference is

\[
4(a^2 - ab + b^2) - (a + b)^2 = 3a^2 - 6ab + 3b^2 = 3(a - b)^2 \geq 0.
\]

3. Let \( A \) and \( B \) be two points on the plane with \( AB = 7 \). What is the set of points \( P \) such that \( PA^2 = PB^2 - 7? \)

Solution. If we let \( K \) be the point on \( AB \) with \( AK = 4, BK = 3 \), then the answer is the line through \( K \) perpendicular to \( AB \). To see this, set \( A = (0, 0) \) and \( B = (7, 0) \). Then the points \( P = (x, y) \) are exactly those satisfying

\[
(x - 0)^2 + (y - 0)^2 = (x - 7)^2 + (y - 0)^2 - 7
\]

which rearranges to \( 9 = -14x + 42, \textit{id est } x = 3. \)

4. The numbers 1, 2, \ldots, 50 are written on a blackboard. We may erase two numbers \( a \) and \( b \), and replace both with \( a + b + 2ab; \) we repeat this operation until only one number remains. Prove that the value of this last number does not depend on how the operations were performed.

Solution. Suppose at some point the numbers on the board are \( x_1, \ldots, x_k \). We claim that the quantity \( (2x_1+1)(2x_2+1) \cdots (2x_k+1) \) does not change. Indeed, this follows from the identity \( 1 + 2(a + b + 2ab) = (1 + 2a)(1 + 2b). \)

Thus, if there is exactly one number \( M \) on the board, it is given exactly by \( 2M + 1 = 3 \cdot 5 \cdots \cdots 101, \) and in particular does not depend on the choice of operations.
5. Show that \( \sin 10^\circ \) is irrational.

**Solution.** One can show the triple-angle identity

\[
\sin(3\theta) = 3\sin \theta - 4\sin^3 \theta.
\]

Thus letting \( x = 2\sin(10^\circ) \) we derive

\[
x^3 - 3x + 1 = 0.
\]

This polynomial has no rational roots (by, say, Rational Root Theorem), hence \( x \) is irrational, whence \( \sin 10^\circ \) is irrational. \( \square \)

6. Let \( c > 0 \) be a positive real number. We define the sequence \((x_n)\) by \( x_0 = 0 \) and

\[
x_{n+1} = x_n^2 + c
\]

for each \( n \geq 0 \). For which values of \( c \) is it true that \( |x_n| < 2016 \) for all \( n \)?

**Solution.** The answer is \( c \leq \frac{1}{4} \).

First, we show that \( c \leq \frac{1}{4} \) all work. Clearly it suffices to prove the result when \( c = \frac{1}{4} \).

In that case, the sequence is defined by \( x_{n+1} = x_n^2 + \frac{1}{4} \). We claim that \( x_n \leq \frac{1}{2} \) for all \( n \). Indeed, this follows by induction, since it is true for \( n = 1 \) and for the inductive step we have

\[
x_{n+1} = x_n^2 + \frac{1}{4} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

Now we show all \( c > \frac{1}{4} \) fail. Assume on the contrary that \( |x_n| < 2016 \) for all \( n \). Then since the real interval \([0, 2016]\) is compact, we must have that \( x_n \) converges to some limit \( x \). This limit must then satisfy

\[
x^2 = x + c.
\]

Thus the discriminant \( 1 - 4c \) is nonnegative, so \( c \leq \frac{1}{4} \) must hold. \( \square \)

7. Let \( ABC \) be a triangle, and let \( X, Y, Z \) be the excenters opposite \( A, B, C \). The incircle of triangle \( ABC \) touches \( BC, CA, AB \) at points \( D, E, F \). Finally, let \( I \) and \( O \) denote the incenter and circumcenter of triangle \( ABC \).

Prove that lines \( DX, EY, FZ, IO \) are concurrent.

**Solution.** The fact that \( DX, EY, FZ \) are concurrent follows from the fact that triangles \( DEF \) and \( XYZ \) are homothetic; indeed, note that \( EF \) and \( YZ \) are both perpendicular to the internal angle bisector of \( \angle BAC \).

Now, to see that the concurrence point lies on \( IO \), note that point \( I \) the orthocenter of triangle \( XYZ \), and \( O \) is the nine-point center of triangle \( XYZ \). Thus line \( IO \) is the Euler line of triangle \( XYZ \) and thus passes through the circumcenter \( S \) of triangle \( XYZ \). But \( I \) is the circumcenter of triangle \( DEF \), hence line \( SI \) passes through the concurrency point. \( \square \)