
*Solution.* Noticing that $25 = 5^2$ and $27 = 3^3$, we have this is

$$5^6 - 3^6 = (5^3 - 3^3)(5^3 + 3^3)$$

$$= (5 - 3)(5^2 + 5 \cdot 3 + 3^2)(5 + 3)(5^2 - 5 \cdot 3 + 3^2)$$

$$= 2 \cdot 49 \cdot 8 \cdot 19$$

$$= 2^4 \cdot 7^2 \cdot 19.$$  

2. Suppose $x$ and $y$ are real numbers satisfying $x + y = 5$. What is the largest possible value of $x^2 + 2xy$?

*Solution.* The quantity in question is $(x + y)^2 - y^2 \leq (x + y)^2 = 25$. Equality occurs when $x = 5$ and $y = 0$, hence the maximum possible value is 25.

3. Let $ABC$ be an acute triangle with orthocenter $H$, circumcenter $O$, and incenter $I$. Prove that ray $AI$ bisects $\angle HAO$.

*Solution.* Without loss of generality, $AB < AC$. It follows that $\angle BAH = 90^\circ - \angle B$, since the extension of $AH$ is perpendicular to $BC$. Moreover, we also have $\angle AOC = 2\angle B$; but since $OA = OC$, this implies $\angle OAC = \frac{1}{2}(180^\circ - \angle AOC) = 90^\circ - \angle B$. So we conclude that $\angle BAH = \angle CAO$. Since $\angle BAI = \angle CAI$ as well, it follows that $\angle HAI = \angle OAI$, which is what we wanted to prove.

4. For which prime numbers $p$ is $p^2 + 2$ also prime? Prove your answer.

*Solution.* The answer is $p = 3$. This indeed works, since $3^2 + 2 = 11$.

Consider any other prime number $p \neq 3$. Then it follows that $p^2 \equiv 1 \pmod{3}$; i.e. that $p$ leaves remainder 1 when divided by 3. Consequently, $p^2 + 2$ is divisible by 3. Since $p \geq 2$, we have $p^2 + 2 \geq 7$ as well, thus $p^2 + 2$ cannot be prime in this case.

5. There is a colony consisting of 100 cells. Every minute, a cell dies with probability $\frac{1}{3}$; otherwise it splits into two identical copies. What is the probability that the colony never goes extinct?

*Solution.* The answer is $1 - \left(\frac{1}{2}\right)^{100}$.

Let $p$ be the probability that a colony consisting of just one cell will survive. Then

$$p = \frac{1}{3} \cdot 0 + \frac{2}{3} \left(1 - (1 - p)^2\right)$$

owing to the fact that when the cell splits in two, the probability both of them go extinct is $(1 - p)^2$. Solving for $p$, we obtain $p = \frac{1}{2}$. 


The colony initially has 100 cells; if we treat these as 100 distinct colonies, we obtain the claimed answer.

6. Let \( H, I, O, \Omega \) denote the orthocenter, incenter, circumcenter and circumcircle of a scalene acute triangle \( ABC \). Prove that if \( \angle BAC = 60^\circ \) then the circumcenter of \( \triangle IHO \) lies on \( \Omega \).

Solution. First, we show that the five points \( B, O, H, I, C \) all lie on a circle. To see this, note that
\[
\angle BIC = 90^\circ + \frac{1}{2} \angle BAC = 120^\circ
\]
\[
\angle BOC = 2 \angle BAC = 120^\circ
\]
\[
\angle BHC = 180^\circ - \angle BAC = 120^\circ.
\]
So, this proves the claim.

Let \( M \) be the midpoint of arc \( BC \) of \( \Omega \) now (not containing \( A \)). Evidently, \( MB = MC \) and \( \angle BMC = 120^\circ \). Since \( OB = OC \) and \( \angle BOC = 120^\circ \) as well, we discover that triangles \( BMO \) and \( CMO \) are actually equilateral triangles, whence \( MB = MO = MC \); i.e. \( M \) is the circumcenter of \( \triangle BOC \). Since \( B, O, H, I, C \) are all concyclic, \( M \) is the circumcenter of \( \triangle IHO \) as well, as desired.

7. Let \( a, b, c \) be positive integers. Prove that it is not possible for \( a^2 + b + c, b^2 + c + a, c^2 + a + b \) to all be perfect squares.

Solution. Without loss of generality we may assume \( \max\{a, b, c\} = a \). Then
\[
a^2 < a^2 + b + c \leq a^2 + 2a < (a + 1)^2.
\]
So, \( a^2 + b + c \) is not a perfect square, because it lies strictly between two perfect squares.

8. Let \( n \) be a fixed positive integer. Initially, \( n \) 1’s are written on a blackboard. Every minute, David picks two numbers \( x \) and \( y \) written on the blackboard, erases them, and writes the number \( (x + y)^4 \) on the blackboard. Show that after \( n - 1 \) minutes, the number written on the blackboard is at least \( 2^{4n^2-4} \).

Solution. We proceed by strong induction \( n \), with the base case \( n = 1 \) being vacuous. For the inductive step, consider the situation in which we have two numbers \( x \) and \( y \) remaining on the blackboard. Suppose the first one was written after \( a-1 \) operations, and the second one was written after \( b-1 \) operations, so that \( (a - 1) + (b - 1) = n - 2 \).

Then by the inductive hypothesis,
\[
x \geq 2^{\frac{4a^2-4}{3}}, \quad y \geq 2^{\frac{4b^2-4}{3}}.
\]
Consequently, using convexity and the bound \( (a + b)^2 \leq 2(a^2 + b^2) \), we have
\[
x + y \geq 2 \cdot 2^{\frac{2(a^2 + b^2) - 4}{3}} \geq 2^{\frac{(a + b)^2 - 1}{3}} = 2^{\frac{n^2 - 1}{3}}.
\]
So \( (x + y)^4 \geq 2^{\frac{4n^2-4}{3}} \) as needed.