Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Contest 3 is due on December 1, 2015.

- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

Submit different problems on different pages as they are graded separately.

- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at http://mathcircle.berkeley.edu for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 3

1. Suppose $x$ and $y$ are real numbers satisfying $x + y = 5$. What is the largest possible value of $x^2 + 2xy$?

2. Let $ABC$ be an acute triangle with orthocenter $H$, circumcenter $O$, and incenter $I$. Prove that ray $AI$ bisects $\angle HAO$.

3. For which prime numbers $p$ is $p^2 + 2$ also prime? Prove your answer.

4. There is a colony consisting of 100 cells. Every minute, a cell dies with probability $\frac{1}{5}$; otherwise it splits into two identical copies. What is the probability that the colony never goes extinct?

5. Let $H$, $I$, $O$, $\Omega$ denote the orthocenter, incenter, circumcenter and circumcircle of a scalene triangle $ABC$. Prove that if $\angle BAC = 60^\circ$ then the circumcenter of $\triangle IHO$ lies on $\Omega$.

6. Let $a$, $b$, $c$ be positive integers. Prove that it is not possible for $a^2 + b + c$, $b^2 + c + a$, $c^2 + a + b$ to all be perfect squares.
7. Let $n$ be a fixed positive integer. Initially, $n$ 1’s are written on a blackboard. Every minute, David picks two numbers $x$ and $y$ written on the blackboard, erases them, and writes the number $(x + y)^4$ on the blackboard. Show that after $n - 1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2 - 4}{3}}$. 