1. Let \( s_1, s_2, \ldots \) be an infinite arithmetic progression of distinct positive integers. Prove that \( s_1, s_2, \ldots \) is also an infinite arithmetic progression of distinct positive integers.

**Solution.** Let \( s_n = an + b \) for some integers \( a \) and \( b \). Then \( s_n = a(an + b) + b = a^2n + (ab + b) \), which is also an arithmetic progression.

2. Is there a polynomial \( P(n) \) with integer coefficients such that \( P(2) = 4 \) and \( P(P(2)) = 7 \)? Prove your answer.

**Solution.** The answer is no. Let \( P(n) = c_nx^n + \cdots + c_0 \). We are given that \( P(2) = 4 \) and \( P(4) = 7 \). The first equation implies that \( c_0 \) is even while the second implies that \( c_0 \) is odd, which is a contradiction.

3. Are there integers \( a, b, c, d \) which satisfy \( a^4 + b^4 + c^4 + 2016 = 10d \)?

**Solution.** The answer is no. Look at the equation in base 5. Observe that \( 0^4 = 0 \), \( 1^4 = 1 = 15 \), \( 2^4 = 16 = 315 \), \( 3^4 = 81 = 3115 \), \( 4^4 = 256 = 20115 \), so each of \( a^4, b^4, c^4 \) must end in 0 or 1 in base 5. On the other hand \( 10d - 2016 \) ends with 4 in base 5. This is impossible.

4. Let \( ABC \) be a triangle and \( P \) a point inside it. Rays \( BP \) and \( CP \) meet \( AC \) and \( AB \) at \( Y \) and \( X \), respectively. Prove that if \( AP \) bisects \( BC \) then \( XY \parallel BC \).

**Solution.** Let \( Q \) be the reflection of \( P \) across \( M \) (with \( M \) the midpoint of \( BC \)). Accordingly, \( BPCQ \) is a parallelogram.

From this, we see that \( \triangle AXP \sim \triangle ABQ \) and \( \triangle AYP \sim \triangle ACQ \), and thus we deduce

\[
\frac{AX}{AB} = \frac{AP}{AQ} = \frac{AY}{AC}
\]

so \( XY \parallel BC \).
5. Yan and Jacob play the following game. Yan shows Jacob a weighted 4-sided die labelled 1, 2, 3, 4, with weights $\frac{1}{4}, \frac{1}{3}, \frac{1}{7}, \frac{1}{8}$, respectively. Then, Jacob specifies 4 positive real numbers $x_1, x_2, x_3, x_4$ such that $x_1 + \cdots + x_4 = 1$. Finally, Yan rolls the dice, and Jacob earns $10 + \log(x_k)$ dollars if the die shows $k$ (note this may be negative). Which $x_i$ should Jacob pick to maximize his expected payoff?

(Here log is the natural logarithm, which has base $e \approx 2.718$.)

**Solution.** Jacob should pick $(x_1, x_2, x_3, x_4) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{8}\right)$. More generally, suppose the weights are $p_1, \ldots, p_4$. Then Jacob’s expected payoff is

$$10 + \sum_{i=1}^{4} p_i \log(x_i) = 10 + \sum_{i=1}^{4} p_i log p_i + \sum_{k=1}^{4} p_i \log \left(\frac{x_k}{p_k}\right).$$

Now, by JENSEN’S INEQUALITY on the concave function $\log x$, we obtain

$$\sum_{i=1}^{4} p_i \log \left(\frac{x_i}{p_i}\right) \leq \log \left(\sum_{i=1}^{4} p_i \cdot \frac{x_i}{p_i}\right) = \log 1 = 0$$

and equality occurs exactly when $\frac{x_1}{p_1} = \frac{x_2}{p_2} = \frac{x_3}{p_3} = \frac{x_4}{p_4}$; that is, when $x_i = p_i$ for every $i$. \(\square\)

6. Let $X = \{1, 2, \ldots, 100\}$. How many functions $f : X \to X$ satisfy $f(b) < f(a) + (b - a)$ for all $1 \leq a < b \leq 100$?

**Solution.** The answer is $\binom{100}{100}$. We claim that the functions are precisely those of the form $f(n) = n + a_n$, where

$$-99 \leq a_{100} < a_{99} < \cdots < a_1 \leq 99$$

is an arbitrary sequence. The answer follows from this.

To see that all functions are of this form, we rewrite the given as $f(b) - b < f(a) - a$, which tells us that $f(100) - 100 < f(99) - 99 < \cdots < f(1) - 1$. Since $f(100) - 100 \geq -99$ and $f(1) - 1 \leq 99$, this shows all functions are of the form claimed above, i.e. that $1 - n \leq f(n) - n \leq 100 - n$.

Similarly, it remains to check that all functions of the form satisfy the conditions. The inequality $f(b) < f(a) + (b - a)$ is immediate. Moreover, it is easy to see that $a_{100} \geq -99$, $a_{99} \geq -98$, and so on, so $1 \leq n + a_n$ holds; similarly, $n + a_n \leq 100$ holds too. Thus $n + a_n$ is indeed an element of $X$. \(\square\)

7. Find, with proof, the largest possible value of

$$\frac{x_1^2 + \cdots + x_n^2}{n}$$

where real numbers $x_1, \ldots, x_n \geq -1$ are satisfying $x_1^3 + \cdots + x_n^3 = 0$.

**Solution.** For any $i$, we have $0 \leq (x_i + 1)(x_i - 2)^2 = x_i^3 - 3x_i^2 + 4$. Adding all of these we deduce that \(\sum_{i=1}^{n} x_i^2 \leq \frac{1}{3} \sum_{i=1}^{n} (x_i^3 + 4) = \frac{2}{3} n\). Equality occurs, for example, when $n = 9, x_1 = \cdots = x_8 = -1$ and $x_9 = 2$. Therefore, the answer is $\frac{4}{3}$. \(\square\)