Frieze patterns

1. In the following table of numbers find a pattern that connects adjacent numbers and allows to extend the table indefinitely to the right and to the left.

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 2 2 3 1 2 4 1 2 2 3 1 2 2 3 1 2 2 3 1 2 2 3 1
3 1 3 5 2 1 7 3 1 3 5 2 1 3 5 2 1 3 5 2 1 3 5 2 1
2 1 7 3 1 3 5 2 1 7 3 1 2 1 7 3 1 2 1 7 3 1 2 1 7 3 1
3 1 2 4 1 2 2 3 1 2 4 1 2 3 1 2 4 1 2 3 1 2 4 1 2
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

**Hint:** Look at the rhombi

```
   b
  a   d
  c
```

Write the rule that you have just found in the box below:

(1)

**Definition.** A frieze is a grid of numbers bounded from above by an infinite row of 0’s, followed by a row of 1’s and satisfying the frieze rule (1). A frieze is called closed if it is also bounded from below by a line of 1’s (followed by a line of 0’s). The number of nontrivial lines in a closed frieze is called the width of the frieze. A frieze is called integral if it consists of integers.

2. Prove that in the frieze

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 2 2 3 1 2 4 1 2 2 3 1 2 2 3 1 2 2 3 1 2 2 3 1
3 1 3 5 2 1 7 3 1 3 5 2 1 3 5 2 1 3 5 2 1 3 5 2 1
2 1 7 3 1 3 5 2 1 7 3 1 2 1 7 3 1 2 1 7 3 1 2 1 7 3 1
3 1 2 4 1 2 2 3 1 2 4 1 2 3 1 2 4 1 2 3 1 2 4 1 2
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

...
with $f_0 = 1$ and $f_1 = a_1$ the following relation holds

$$f_i = a_i f_{i-1} - f_{i-2}. \tag{2}$$

**Hint.** Check the statement for $i = 2$ and then use the row of $g_i$’s and induction on $i$.

3. Use relations (1) and (2) to show that if all the numbers $a_i$ are positive integers than the frieze is integral and all it’s entries are positive.

4. Prove that every line of a closed frieze of width $n - 3 \geq 1$ is $n$-periodic.

**Hint.** Plug $i = n - 1$ into relation (2) and then consider a diagonal starting with $f_{n-3}$ and going North-East.

5. Prove that a closed integral frieze with the first nontrivial line $(a_i)$ has $a_j = 1$ for some $j$.

**Hint.** Assume that there is no such $j$. Show that $f_i > f_{i-1}$ by induction on $i$.

6. Consider a closed integral frieze $F$ of width $w$ and choose $j$ such that $a_j = 1$. By problem 4, the frieze $F$ is periodic with period $n = w + 3$. Let $F'$ be the $n - 1$-periodic frieze with the first nontrivial line

$$\ldots, a_{j-1}, a_j, a_{j+1}, a_{j+2}, \ldots \rightarrow \ldots, a_{j-2}, a_{j-1} - 1, a_{j+1} - 1, a_{j+2}, \ldots$$

Prove that the frieze $F'$ is a closed integral frieze of width $w - 1$. Note that you can construct the frieze $F$ from the frieze $F'$ by reverting the process.

**Hint.** Consider the diagonal $f'_i$ for the frieze $F'$ and show that $f'_i = f_i$ if $i \leq j - 2$ and $f'_i = f_{i+1}$ if $i \geq j - 1$.

7. Consider a triangulated $n$-gon. For every vertex $v_i$ of the $n$-gon, let $a_i$ be its number of adjacent triangles; this yields an $n$-periodic sequence $(a_i)$. Taking this sequence as the first nontrivial line, we define the frieze corresponding to the triangulation. Draw the $n$-gon and its triangulation corresponding to the frieze in problem 1.

8. Prove the following theorem by Conway and Coxeter.

**Theorem.**

(i) The frieze corresponding to a triangulation of an $n$-gon is a closed integral frieze of width $n - 3$.

(ii) Every closed integral frieze of width $n - 3$ corresponds to some triangulation of an $n$-gon.

**Hint.** Consider $n = 3$ and then prove the theorem using induction on $n$.