Rigid motions, symmetry and crystals.
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Berkeley Math Circle, October 31, 2004

Rigid motions on a plane.
A rigid motion is a map of a plane to itself which preserves distances and angles.
1. Show that a parallel translation, a central symmetry, a rotation and a reflection are rigid motions.
2. Show that a composition of two rigid motions is a rigid motion.
3. Show that a composition of two central symmetries is a parallel translation.
4. Show that a composition of two reflections is a parallel translation or a rotation.
5. What is a composition of two rotations with different centers?
6. Let $ABC$ and $A'B'C'$ be congruent triangles. Show that there is exactly one rigid motion which send $A$ to $A'$, $B$ to $B'$ and $C$ to $C'$.
7. Show that any rigid motion is a translation, a rotation, a reflection or a composition of a reflection and a translation along the line of reflection.

Rigid motions in geometric problems.
Rigid motions and symmetry are very useful in some geometric problems. Here are some examples.

1. Given an angle $\angle ABC$ and a point $D$ inside this angle. Construct a segment with the endpoints on the sides of the angle $\angle ABC$ and the midpoint $D$.
2. There is a regular polygon with 10 vertices. Two players play the following game. Each player in his turn draws a diagonal which does not cut previously drawn diagonals. The player who can not make a move looses. Who can always win in this game?
3. Given a line $l$ and points $A$ and $B$ on the same side of $l$. Find the point $X$ on $l$ such that $AX + XB$ is the smallest.
4. Given an acute triangle $ABC$. Find points $P, Q$ and $R$, one on every side of this triangle, such that the triangle $PQR$ has the smallest perimeter.
5. On the sides of a triangle $ABC$ the squares $ABMN$ and $BCPQ$ are constructed (outside of the triangle). Show that the centers of the squares and the midpoints of $AC$ and $QM$ are vertices of another square.
6. On the sides of a triangle the equilateral triangles are constructed (outside of the triangle). Show that the centers of these equilateral triangles form another equilateral triangle.
7. Find a point $X$ inside the triangle $ABC$ such that $AX + BX + CX$ is the smallest.

Rigid motions in space.
One can define rigid motion in space in the same way as on a plane.
1. A parallel translation, a reflection in a plane and a rotation about some axis are rigid motions.
2. Let $F$ be a rigid motion which does not move a point $P$. Then $F$ is either a rotation or a reflection composed with some rotation (may be on 0 degrees).
3. Any rigid motion in space is a rotation, a parallel translation, a reflection or a composition of these motions.

**Groups of symmetries.**

Let $G$ be some set of rigid motions (on a plane or in space) satisfying two properties

1. if a rigid motion is in $G$ then its inverse is also in $G$;
2. a composition of two motions from $G$ is again in $G$.

We call such $G$ a *group*. A typical example of a group is the set of all rigid motions preserving a given geometric figure. A group of symmetries of a rectangle contains two reflections, a central symmetry and the identity map. So it has four elements.

1. If the group of symmetries of a plane figure contains more than one central symmetry, then it has infinitely many central symmetries.
2. Show that a polygon has at most one center of symmetry.
3. Given a hexagon such that any two opposite sides are parallel and congruent. Show that this hexagon is centrally symmetric.
4. Describe all quadrilaterals with group of symmetries of 4 elements.
5. Find the number of symmetries of a parallelogram, a square, an equilateral triangle, a regular tetrahedron, a cube and a dodecahedron.
6. Show that the group of symmetries of a cube contains a group of symmetries of a tetrahedron. (Hint: inscribe a regular tetrahedron in a cube.)
7. List the angles of rotations for all rotations which are symmetries of a dodecahedron?
8. Show that any group of rigid motions with finitely many elements fixes a point.
9. Show that any finite group of rigid motions on plane is the group of symmetries of a regular polygon or of some quadrilateral, or just a group of rotations on the multiples of the angle $\frac{360c}{n}$.

**Crystals and crystallographic groups.**

A set $M$ of points on a plane (in space) is called *regular* if

1. every circle (ball) contains finitely many points from $M$;
2. every circle (ball) with sufficiently large radius contains a point of $M$;
3. for any two points $x$ and $y$ from $M$ there is a rigid motion from the group of symmetries of $M$ which maps $x$ to $y$.

It is clear that a regular set $M$ has a large (infinite) group of symmetries $G$. A group $G$ of rigid motions is called a *crystallographic group* if by applying all motions from $G$ to some point $x$ one gets a regular set $M$.

1. Let $G$ be a crystallographic group on a plane (in space) and $T$ be the set of all translations from $G$. Show that $T$ is a group.
2. Show that $T$ can be obtained starting with two (three) parallel translations on a plane (in space) by applying compositions and taking inverse.
3. Let $x$ be a point of a regular set $M$ and $G_x$ be the set of symmetries from $G$ preserving $x$. Show that $G_x$ is a finite group.
4. List all $G_x$ possible for crystallographic groups $G$ on a plane. You should get 10 different groups.

5. Describe all crystallographic groups on a plane. You should get 17 different groups.

6. Think about problems 4 and 5 in space. There are 32 possible $G_x$ and 230 different crystallographic groups! (Actually in crystallography all these groups have names. If you take exam in crystallography, you have to know them all.)

7. Let $G$ be a crystallographic group. There exists a polygon (a polyhedron for the space) such that its images under the symmetries from $G$ cover the whole plane (space) without overlapping.

   Place several atoms inside this polyhedron, proliferate them along the space. You get a picture similar to how atoms are placed in real crystals.