Maxima and Minima *Without* Calculus

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1 Background

The title of this talk is the same as the title of the book **[1]** by Ivan Niven published by the Mathematical Association of America in the Dolciani Series. It is written as exposition at a level for undergraduate students and I recommend it to you if you want to look more closely at the subject of the talk today and related topics. My list of references at the end of the book is not complete. There is a vast list of articles and books compiled in this book by Niven and you should go there first. Although I knew that many problems in calculus had been solved before calculus was invented, I had no idea of the wide variety of problems that one sees in every calculus book among the exercises that can be solved with just a few techniques. I remember solving them as a student and now I assign them as a teacher. However, I do point out to *my* classes that there are elementary methods that often are easier to implement. Professor Ogilvy has said "these are 'trick methods', each applying solely to its own problem. Usually they cannot be extended, lacking the great generality of the analytic (calculus) methods." For example:

Find the coordinates where $f(x) = \dfrac{9x^2\sin^2 x + 4}{x\sin x}$ has a relative maximum on $(0, \pi)$.

Niven on the other hand agrees that while calculus is good "for solving *some* problems in maxima and minima, the method is not universal. There are many problems that are awkward, if not impossible, to solve with elementary calculus ... Thus we follow a simple maxim: If a problem can be solved more simply with calculus, leave it to calculus." Some examples:

Find the minimum value of $f(x) = \dfrac{9x^2\sin^2 x + 4}{x\sin x}$ on $(0, \pi)$.

Find the quadrilateral with the largest area with a given perimeter.

2 Preliminaries

Here several interesting and useful facts about triangles. Let s denote the semiperimeter of triangle ABC , α, β, γ the angles, a, b, c the opposite sides, and K the area of the triangle.

1. $K = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$.
2. $K = \sqrt{s(s-a)(s-b)(s-c)}$. (Heron's formula)

3. $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$ (Law of Cosines).

4. $2R = \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ (Law of Sines). (R is the radius of the circumcircle)

See [**2**] for an elementary algebra proof of Heron's formula.
Let s denote the semiperimeter of quadrilateral $ABCD$,
$\alpha, \beta, \gamma, \delta$ the angles, $AB = a, BC = b, CD = c, DA = d$ and Q the area.

1. $Q = \frac{1}{2}(a)(b)\sin\beta + \frac{1}{2}(d)(c)\sin\delta$.

2. $Q = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(\cos(\beta+\delta)+1)}$

3 Arithmetic Mean-Geometric Mean Inequality

In both algebra and geometry, one will encounter the fact that for any positive real numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab}$, or in words, the arithmetic mean (average) is greater than or equal to the geometric mean. This can be generalized to n numbers in the following way.

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \geq (a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}}$$

with equality for $a_1 = a_2 = a_3 = \cdots = a_n$. See problem #23 in the notes to Ted Alper's talk in November on Mathematical Induction [**3**]. See the notes to Bjorn Poonen's talk on Inequalities [**4**] for a much more extensive examination of the topic of inequalities with a great set of problems to work on. I didn't learn much about mathematics in the classes that I took for three and a half years to get my degree in mathematics. I found out a great deal about mathematics later, from a series of books called the *New Mathematical Library* which is now being published by the Mathematical Association of America. The first book in the series is *Numbers: Rational and Irrational* by Ivan Niven. The fifteenth book in the series is *Mathematics of Choice: How to Count without Counting* also by Ivan Niven. For a gentle but thorough introduction to the subject of inequalities I recommend two more volumes in the series; *Introduction to Inequalities* [**5**] and *Geometric Inequalities* [**6**]. For this talk, the case of $n = 3$ will be sufficient, so we will look at an idea from Paul Zeitz's book on problem solving [**7**]. As every student from China, whom I have received into my classes seems to know when they arrive

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$$

From this we can now deduce the case of $n = 3$ of the AM-GM inequality.

4 Problems

1. Find the least value of $\dfrac{x}{y} + \dfrac{3y}{z} + \dfrac{9z}{x}$ over all positive real numbers x, y,and z.

2. For any positive constant c, find the maximum value of $xy(c - x - y)$ over all positive numbers x and y.

3. If a is any positive constant, find the minimum value of $x^2 + \dfrac{a}{x}$.

4. Find the maximum value of the product $xy(72 - 3x - 4y)$ for positive x and y.

5. Find the dimensions of the box of maximum volume that has one corner at the origin, three sides that contain that corner lying in the three coordinate planes and the opposite corner lying on the plane $2x + 3y + 4z = 12$ in the first octant. (*How to Ace* the rest of *Calculus*)

6. Find the smallest value of 5x + 16/x + 21.

7. Find the maximum and minimum values, if any, of the function $f(x) = \sqrt{100 + x^2} - x$ over the domain $x \geq 0$.

8. Find the least value of the sum $x^2 + 4x + 4/x + 1/x^2$ over positive real numbers x.

9. Find the least value of $f(x) = \dfrac{(x + 10)(x + 2)}{x + 1}$.

10. Find the maximum value of x^2y if x and y are restricted to positive real numbers satisfying $6x + 5y = 45$.

11. For any positive constant a, find the maximum of $\dfrac{x}{x^2 + a}$ all positive x.

12. For any positive constant a, find the maximum of $\dfrac{x^2}{x^3 + a}$ all positive x.

13. A manufacturer makes aluminum cups of volume 16 cubic inches in the form of right circular cylinders. Find the dimensions that use the least material. (OHS, Jan 2002)

14. Minimize the expression $6x + 24/x^2$ over positive numbers x.

15. Find the maximum value of $\dfrac{12(xy - 4x - 3y)}{x^2y^3}$ with x and y positive.
    (Hint: Use AM-GM with $n = 4$.)

16. Find the least value of $xy + 2xz + 3yz$ for positive numbers x, y, z, satisfying $xyz = 48$.

17. Multiply $(x + y + z)\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$ and find the minimum value of the product. Hence find the least value of $x^{-1} + y^{-1} + z^{-1}$ over the positive real numbers x, y, z having a constant sum.

18. Show that among all the triangles of a given perimeter, the equilateral triangle has the largest area.

19. Show that among all the quadrilaterals of a specified perimeter, the square has the largest area.

20. Show that a quadrilateral inscribed in a circle has a larger area than any other quadrilateral with sides of the same lengths in the same order.

21. Find the length of the longest ladder that can be moved around a right-angle corner from a corridor of width a to a corridor of width b. (*Calculus* by Larson, et.al.)

22. Two posts, one 12 feet high and the 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. How long a wire is needed? (*Calculus* by Larson, et.al.)

23. Given a line segment AB and a line ℓ not intersecting the given line segment, find the point P on ℓ such that the segment AB subtends the greatest angle at P.

24. AB is a diameter of a circle of radius 1. C and E are distinct points on the circle and on the same side of AB. Parallel chords CD and EF cut AB at a 45° angle, at points P and Q, respectively. Prove that $PC \cdot QE + PD \cdot QF < 2$. (China National 1981)

25. If the sum of the lengths of six edges of a trirectangular tetrahedron $PABC$ (i.e. $\angle APB = \angle BPC = \angle CPA = 90°$) is S, determine its maximum volume.
(Fifth USAMO 1976)

5 References

1. Ivan Niven, *Maxima and Minima Without Calculus*, Mathematical Association of America, 1981.

2. H. Jacobs, *Geometry*, W.H.Freeman and Company, 1987.

3. Ted Alper, *Mathematical Induction*, Berkeley Math Circle, November 18, 2001.

4. Bjorn Poonen, *Inequalities*, Berkeley Math Circle, November 7, 1999.

5. E. Beckenback, R. Bellman, *An Introduction to Inequalities*, Mathematical Association of America, 1961.

6. N. Kazarinoff, *Geometric Inequalities*, Mathematical Association of America, 1961.

7. Paul Zeitz. *The Art and Craft of Problem Solving.* John Wiley & Sons, Inc., 1999.

8. G. Wentworth, D.E. Smith, *Plane Geometry*, Ginn and Company,1913.

9. Morris Levenson, *Maxima and Minima*, Macmillan Company, 1967.

10. C.S. Ogilvy, *A Calculus Notebook*, Prindle, Weber and Schmidt, 1968.

11. G. Polya, *Mathematics and Plausible Reasoning, Volume 1*, Princeton University Press, 1954.

6 Two more problems

1. Find the least value of the expression $(x + y)(y + z)$, given that x, y, z are positive numbers satisfying the equation $xyz(x + y + z) = 1$. (U.S.S.R. Olympiad 1989 #21)

2. Find the minimum value of $(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v}\right)^2$ for $0 < u < \sqrt{2}$ and $v > 0$.
(1984 Putnam Exam B2 and BAMM 1999 #16)

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us