BAMO math circle, March 25,2001
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1. A grasshopper can jump p or q inches right or left on the line. Find all points on the line the grasshopper can reach starting from the origin.

2. Prove that the minimal positive integer d of the form $d = mp + nq$ coincides with the Greatest Common Divisor of p and q, and that $GCD(p, q)$ can be found by the following *Euclidean algorithm:*

Divide p by $q > 0$ with the remainder r: $p = lq + r$ where $q > r \geq 0$. If $r > 0$, proceed with q, r instead of p, q. If $r = 0$, stop. The last non-zero remainder d equals $GCD(p, q)$.

3. Diophantine equation $ax + by = c$ has a solution (x, y) if and only if c is divisible by $d = GCD(a, b)$. If (x_0, y_0) is a solution, then any other solution can be written in form $x = x_0 + sb/d, y = y_0 - sa/d$ for some integer s.

4. The equation $x^2 - y^2 = 10$ does not have integer solutions.

5. Prove that the equation $x^2 + y^2 = z^2$ has infinitely many non-proportional integer solutions.

6. Find all integer solutions of the equation $x^2 - y^2 = 2001$.

7. Obtain a formula for all rational solutions of $x^2 + y^2 = 1$. For this consider an inversion on the plane with center at the point $(0, 1)$ and radius 2. Show that the inversion maps the x-axis onto the unit circle with center at the origin without the point $(0, 1)$. Prove that a rational point on x-axis moves to a point with rational coordinates on the circle.

8. Prove that sums, differences, products and ratios of numbers of the form $a + b\sqrt{2}$, where a, b are rational, also have this form. Is it true if a, b ought to be integer?

9. Theorem: the equation $x^2 - 2y^2 = 1$ has infinitely many integer solutions. *Hint:* Prove that if (a, b) and (c, d) are two solutions then $(ac + 2bd, ad + bc)$ is a solution too.

10. Put $N(a + b\sqrt{2}) = (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$. Show that $N(AB) = N(A)N(B)$.

11. Find all numbers $A = a + b\sqrt{2}$ with integer a, b such that $N(A) = 1$.

Homework

1. A Heffelump is a chess piece which moves like Knight but with p steps in one direction and q steps in the perpendicular direction. Determine for which p and q the Heffelump, starting from one cell on the infinite chess board, can reach any other cell.

2. Solve the following Diophantine equations:
(a)$161x + 7y = 1$;
(b)$4x + 13y = 1$;
(c)$3x + 5y = 41$;
(d)$xy = x + y$;
(e)$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

3. Show that the equations below do not have integer solutions:
(a)$x^4 + y^4 = 10$;
(b)$x^2 + y^2 = 7z^2$;
(c)$x^2 - 3y^2 = 2$;
(d)$x^2 - 3y^2 = 7$;
(e)$x^2 - 2y^2 = 3$

4. Write recurrent formulae for all integer solutions of the following equations:
(a)$x^2 - 2y^2 = 2$;
(b)$x^2 - 3y^2 = 1$;
(c)$x^2 - 3y^2 = 6$.

5. Show that if p is prime then the equation $x^2 - py^2 = 1$ has infinitely many solutions.

(a) Show that one can find an integer b such that $x^2 - py^2 = b$ has infinitely many solutions.

(b) Let (x_1, y_1) and (x_2, y_2) be two solutions of the equation from (a), x_1 and x_2 have the same remainder when divided by b and y_1 and y_2 have the same remainder when divided by b. Let $Z[\sqrt{p}]$ be the set of all numbers $a + b[\sqrt{p}]$ with integer a and b. Prove that $\frac{x_1 + y_1\sqrt{p}}{x_2 + y_2\sqrt{p}}$ belongs to $Z[\sqrt{p}]$ and has norm 1.

(c) Using (b) obtain infinitely many solutions for $x^2 - py^2 = 1$.