

BERKELEY MATH CIRCLE 2000-2001

Vectors - Applications to Problem Solving

ZVEZDELINA STANKOVA-FRENKEL
MILLS COLLEGE

1. WELL-KNOWN FACTS

- (1) Let A_1 and B_1 be the midpoints of the sides BC and AC of $\triangle ABC$. Prove that

$$(a) \quad \overrightarrow{AA_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}); \quad (b) \quad \overrightarrow{B_1A_1} = \frac{1}{2}\overrightarrow{AB}.$$

- (2) Let A_1 and B_1 be the midpoints of the sides BC and AD of quadrilateral $ABCD$. Prove that

$$\overrightarrow{B_1A_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC}).$$

Note that Exercise 2 generalizes Exercise 1: for part (a) let D coincide with A ; for part (b) let D coincide with C .

- (3) Consider vector \overrightarrow{XY} , and draw two “paths” of vectors $\vec{v}_1, \dots, \vec{v}_n$ and $\vec{w}_1, \dots, \vec{w}_m$ such that each path starts at point X and ends at point Y . Prove that

$$\overrightarrow{XY} = \frac{1}{2}(\vec{v}_1 + \dots + \vec{v}_n + \vec{w}_1 + \dots + \vec{w}_m).$$

Note that Exercise 3 further generalizes Exercise 2.

- (4) Let f be any of the following transformations of the plane: a rotation, a translation, a homothety, a reflection, or a composition of the above. Let \vec{v} and \vec{w} be two vectors in the plane. Prove that

$$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}).$$

Definition. A *distance-preserving* transformation f of the plane is a transformation which preserves all pairwise distances, i.e. for any two points A and B in the plane we have that the distance between A and B is the same as the distance between their images $f(A)$ and $f(B)$ under the transformation: $|AB| = |f(A)f(B)|$.

- (5) Check that rotations, translations and reflections are distance-preserving transformations, but homotheties are *not* except in the cases of the identity transformation and central symmetries (both of which are special cases of homotheties.) Further, prove that any distance-preserving transformation of the plane is a composition of a translation and a rotation, or of a translation and a reflection.

2. CENTROID AND LEIBNITZ THEOREM

Definition. Given points A_1, A_2, \dots, A_n (in the plane or in space), the *centroid* G of these points is the unique point which satisfies

$$\vec{GA}_1 + \vec{GA}_2 + \dots + \vec{GA}_n = \vec{0}.$$

Note that the *medicenter* of any $\triangle ABC$ is the centroid of the vertices A, B, C .

- (6) Prove that for any points A_1, A_2, \dots, A_n there exists a unique centroid G as defined above.

- (7) Let G be the centroid of A_1, A_2, \dots, A_n , and X - an arbitrary point. Prove that

$$\vec{XA}_1 + \vec{XA}_2 + \dots + \vec{XA}_n = n \vec{XG}.$$

Note that for $X = G$, this reduces to the definition of the centroid G .

- (8) (Leibnitz) Let G be the medicenter of $\triangle ABC$, X - an arbitrary point. Prove that

$$XA^2 + XB^2 + XC^2 = 3XG^2 + GA^2 + GB^2 + GC^2.$$

Generalize to an arbitrary polygon $A_1A_2\dots A_n$ with centroid G :

$$\sum_{i=1}^n XA_i^2 = nXG^2 + \sum_{i=1}^n GA_i^2.$$

- (9) Let H be the orthocenter of $\triangle ABC$ and let R be its circumradius. Prove that

$$HA^2 + HB^2 + HC^2 \geq 3R^2.$$

When is equality obtained?

- (10) (84.49) Let $\triangle ABC$ with medicenter G be inscribed in a circle of center O . Point M lies inside the circle with diameter OG . Lines AM, BM and CM intersect the circumcircle again in points A', B' and C' , respectively. Prove that the area of $\triangle ABC$ is not greater than the area of $\triangle A'B'C'$.
- (11) (G260) A point M and a circle k are given in the plane. If $ABCD$ is an arbitrary square inscribed in k , prove that the sum $MA^4 + MB^4 + MC^4 + MD^4$ is independent of the positioning of the square. Replace now the square by a regular n -gon $A_1A_2\dots A_n$. Let $S_m = \sum_i MA_i^m$. For what natural m is S_m independent of the position of the n -gon (still inscribed in k)?
- (12) (G270) Points A_1, A_2, \dots, A_n ($n \geq 3$) lie on a circle with center O . Drop a perpendicular through the centroid of every $n - 2$ of these points towards the line determined by the remaining two points. Prove that the $\binom{n}{2}$ thus drawn lines are all concurrent.
- (13) (G271) Points A_1, A_2, \dots, A_n ($n \geq 2$) lie on a sphere. Drop a perpendicular through the centroid of every $n - 1$ of these points towards the plane, tangent to the sphere at the remaining n -th point. Prove that the n drawn lines are all concurrent.

3. ROTATIONS AND SIMILARITIES

- (14) Let $\rho(O, \alpha)$ be a rotation about angle α and centered at point O . Let g be a line in the plane, and g' be its image under the rotation. Let M and M' be the feet of the perpendiculars dropped from O to g and g' , and let M_1 be the intersection point of g and g' . Prove that
- (a) the angle between g and g' equals α .
 - (b) one can map point M into M_1 by composing a rotation $\rho_1(O, \alpha/2)$ and a homothety $h(O, 1/(\cos \frac{\alpha}{2}))$, i.e. a *similarity* $s(O, \alpha, 1/(\cos \frac{\alpha}{2}))$.
- (15) (G262) $\triangle ABC$ is rotated to $\triangle A'B'C'$ around its circumcenter O by angle α . Let A_1, B_1 and C_1 be the intersection points of lines BC and $B'C'$, CA and $C'A'$, and AB and $A'B'$, respectively. Prove that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar, and find the ratio of their sides.
- (16) (G263) The quadrilateral $ABCD$ is inscribed in a circle k with center O , and the quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB , $B'C'$ and BC , $C'D'$ and CD , and $D'A'$ and DA . Prove that $A_1B_1C_1D_1$ is a parallelogram.
- (17) (G264) In quadrilateral $ABCD$, the diagonals intersect in point O . Quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB , $B'C'$ and BC , $C'D'$ and CD , and $D'A'$ and DA . Prove that $A_1B_1C_1D_1$ is cyclic if and only if $AC \perp BD$.
- (18) Let $A_1A_2A_3A_4$ be an arbitrary cyclic quadrilateral. Denote by H_1, H_2, H_3 and H_4 the orthocenters of $\triangle A_2A_3A_4$, $\triangle A_3A_4A_1$, $\triangle A_4A_1A_2$ and $\triangle A_1A_2A_3$, respectively. Prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.
- (19) (Kazanluk'97 X) Point F on the base AB of trapezoid $ABCD$ is such that $DF = CF$. Let E be the intersection point of the diagonals AC and BD , and O_1 and O_2 be the circumcenters of $\triangle ADF$ and $\triangle BCF$, respectively. Prove that the lines FE and O_1O_2 are perpendicular.
- (20) (Bulgaria'00) Point D is a midpoint of the base AB of the acute isosceles $\triangle ABC$. Let $E \neq D$ be an arbitrary point on the base, and O - the circumcenter of $\triangle ACE$. Prove that the line through D perpendicular to DO , the line through E perpendicular to BC , and the line through B parallel to AC intersect in one point.

4. COMPOSITIONS OF ROTATIONS

- (21) Prove that the composition of two rotations $\rho_1(O_1, \alpha_1)$ and $\rho_2(O_2, \alpha_2)$ about different centers O_1 and O_2 is:
- (a) rotation if $\alpha_1 + \alpha_2 \neq k\pi$ ($k \in \mathbb{Z}$);
 - (b) translation if $\alpha_1 + \alpha_2 = 2k\pi$ ($k \in \mathbb{Z}$);
 - (c) central symmetry if $\alpha_1 + \alpha_2 = (2k + 1)\pi$ ($k \in \mathbb{Z}$).
- (22) (G267) On the sides of a convex quadrilateral draw externally squares. Prove that the quadrilateral with vertices the centers of the squares has equal perpendicular diagonals.

- (23) (G268) Given two equally oriented equilateral triangles AB_1C_1 and AB_2C_2 with centers O_1 and O_2 , respectively, let M be the midpoint of B_1C_2 . Prove that $\triangle O_1MB_2 \sim \triangle O_2MC_1$.
- (24) (G269) A hexagon $ABCDEF$ is inscribed in a circle of radius r so that $AB = CD = EF = r$. Let the midpoints of BC, DE, FA be L, M, N respectively. Prove that $\triangle LMN$ is equilateral.
- (25) (Napoleon) If three equilateral triangles ABC_1, BCA_1 and CAB_1 are constructed off the sides of $\triangle ABC$, show that the centers of these equilateral triangle form another equilateral triangle. Prove also that AA_1, BB_1 and CC_1 are concurrent and have same lengths. Can you identify the medicenter of $\triangle O_1O_2O_3$ with some distinguished point of $\triangle ABC$?

5. METRIC RELATIONS AND GEOMETRIC LOCI OF POINTS

- (26) (Stuard) Prove that if point D lies on the side BC of $\triangle ABC$, and $BC = a, CA = b, AB = c, BD = m, CD = n, AD = d$, then $d^2a = b^2m + c^2n - amn$. In particular, for the median AM in $\triangle ABC$ we have

$$4AM^2 = 2(b^2 + c^2) - a^2.$$

- (27) (Kazanluk'95 X) Given $\triangle ABC$ with sides $AB = 22, BC = 19, CA = 13$,
- If M is the medicenter of $\triangle ABC$, prove that $AM^2 + CM^2 = BM^2$.
 - Find the locus of points P in the plane such that $AP^2 + CP^2 = BP^2$.
 - Find the minimum and maximum of BP if $AP^2 + CP^2 = BP^2$.
- (28) (G272) Given $\triangle ABC$, find the locus of points M in the plane such that $MA^2 + MB^2 = MC^2$.
- (29) (G273) Given tetrahedron $ABCD$, find the locus of points M in such that $MA^2 + MB^2 + MC^2 = MD^2$. How about $MA^2 + MB^2 = MC^2 + MD^2$?
- (30) (UNICEF'95) Given a fixed segment AB and a constant $k > 0$, find the locus of points C in the plane such that in $\triangle ABC$ the ratio of some side to the altitude dropped to this side equals k .
- (31) (UNICEF'95) We are given $\triangle ABC$ in the plane. A rectangle $MNPQ$ is called *circumscribed* around $\triangle ABC$ if on each side of the rectangle there is at least one vertex of the triangle. Find the locus of all centers O of the rectangles $MNPQ$ circumscribed around $\triangle ABC$.
- (32) (84.22) The orthogonal projections of a right triangle onto the planes of two faces of a regular tetrahedron are themselves regular triangles of sides 1. Find the perimeter of the right triangle.
- (33) (84.42) Given a pyramid $SABCD$ whose base is the parallelogram $ABCD$. Let N be the midpoint of BC . A plane γ moves in such a way that it always intersects lines SC, SA and AB in points P, Q and R and

$$\frac{\overline{CP}}{\overline{CS}} = \frac{\overline{SQ}}{\overline{SA}} = \frac{\overline{AR}}{\overline{AB}}.$$

Point M on line SD is such that line MN is parallel to plane γ . Find the locus of points M as γ runs over all possible positions.