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Problems With Parameters.

Problems of the type discussed often appear on the entrance exams to Russian and Ukrainian universities. Most of the exercises in this handout are taken from the exams given at Moscow State University (MSU) and other universities (see the last page for complete list and abbreviations).

What is a parameter? Examples from algebra course.

Linear function: $y = kx+b$ (here x and y are variables, k and b - parameters);

Quadratic equation: $ax^2 + bx + c = 0$ (x- variable, a, b, c - parameters, $a \neq 0$);

Polynomial equations of degree n: $\sum a_i x^i = 0$ (x - variable, a_i - parameters, $a_n \neq 0$).

Simple ones, "branching".

1. Solve $ax = 1$

   Solution: If $a \neq 0$; $x = \frac{1}{a}$, if $a = 0$ - no solutions.

2. Solve $|x^2 - 1| + |a(x - 1)| = 0$

   Solution: This is equivalent to

   $$x^2 - 1 = 0, a(x - 1) = 0.$$

   Answer:If $a \neq 0$ then $x = 1$; if $a = 0$, $x = \pm 1$

3. For which a does equation $\frac{x^2-ax+1}{x+3} = 0$ has exactly one solution.

   Solution: Equivalently,

   $$x^2 - ax + 1 = 0, x \neq -3$$

   Either discriminant is 0 **or** one of the roots of the quadratic is -3. In the first case, $a = +- 2$ and roots of the quadratic are different from -3. In the second case we have $a = -\frac{10}{3}$ and the second root is different from -3.

   Answer: $a = \pm 2, a = -\frac{10}{3}$

   To see just how messy it can get try this one:

4. (MEPHI) Solve $\tan 4x - \tan(2x - \frac{\pi}{4}) = c - 1$

   Answer: If $c < 0$ or $c \geq 2$ then $x = +- \frac{1}{2} \arctan \sqrt{\frac{c-2}{c}} + \frac{\pi k}{2}$, $k \in \mathrm{Z}$; if $c = 0$, then $x = \frac{\pi}{4} + \frac{\pi n}{2}$, $n \in \mathrm{Z}$; for other c - no solutions.

Analytic methods and tricks: properties of functions, trig substitutions, parameter as a variable etc.

1. For which a and b the following system has solutions:

$$2a + \frac{2b^2}{a} = x(1 + \frac{b}{a}) + y(1 - \frac{b}{a}),$$
$$\frac{2a}{b} - 2ab = x(\frac{1}{b} - a) + y(\frac{1}{b} + a).$$

   Find these solutions.

   Solution: This is linear with respect to a and b. Let's solve for them. Equivalent system is

   $2a^3+2b^2a = xa^2+xba+ya^2-yab, 2a-2ab^2 = x-xab+y+yab, a \neq 0, b \neq 0$

   Adding the equations we get:

   $2a(a^2+1) = x(a^2+1)+y(a^2+1), 2a-2ab^2 = x-xab+y+yab, a \neq 0, b \neq 0$

   It follows that $a = \frac{x+y}{2}, b = \frac{x-y}{2}$ and $x = a+b, y = a-b$ This example is a bit "artificial"; however the tactic of solving for parameter is often useful.

2. (MSU) Find all a for which $\sqrt{a + \sqrt{a + \sin x}} = \sin x$ has solutions.

   Answer: $-\frac{1}{4} \leq a \leq 0$.

3. For each non-negative a solve $a^3x^4 + 6a^2x^2 - x + 9a + 3 \geq 0$.

   Answer: If $a = 0$, then $x \leq 3$; if $0 < a < \frac{1}{12}$, then $x \leq \frac{1-\sqrt{1-12a}}{2a}$ or $x \geq \frac{1-\sqrt{1-12a}}{2a}$; if $a \geq \frac{1}{12}$ then x -any.

4. Solve $\sqrt{5 - x} = x^2 - 5$ *Solution hint:* Solve for 5.

5. Solve $x(2x^2 - 1)\sqrt{1 - x^2} = a$

   Solution: As $|x| \leq 1$, we can substitute $x = \cos\alpha$, $\alpha \in [0, \pi]$, so that $\sin\alpha \leq 0$ we get $\cos\alpha\cos 2\alpha\sin\alpha = \alpha$, so that $\sin 4\alpha = 4a$. Now it's trivial. If $|a| > 0$ - no solutions. Otherwise $\alpha = (-1)^k \frac{1}{4}\arcsin 4a + \frac{\pi k}{4}$. If $-\frac{1}{4} \leq a < 0$ than this is in $[0, \pi]$ for $k = 1, 2, 3, 4$. If $a = 0$, than $k = 0, 1, 2, 3, 4$. If $0 < \alpha \leq \frac{1}{4}$ then $k = 0, 1, 2, 3$. Writing out the answer is an exercise left for the reader.

6. For which a does $3xy - 4x^2 < a(x^2 + y^2)$ has solutions?

   Answer:$a > -\frac{9}{2}$

   In solving inequalities it is useful to "isolate" the parameter and study the function with which it is compared. Also note, that if we are asked for something other than exact and complete solution we should concentrate on what we are asked.

7. (MSU) Find the maximal value of b for which the inequality

$$\sqrt{b^5}(8x - x^2 - 16) + \frac{\sqrt{b}}{8x - x^2 - 16} \geq -\frac{2}{3}b|\cos \pi x|$$

has at least one solution. *Solution:* For $b = 0$ it does. If $b \neq 0$ We have equivalently

$$\sqrt{b^5}(x-4)^2 + \frac{\sqrt{b}}{(x-4)^2} \leq \frac{2}{3}b|\cos \pi x|.$$

Left hand side is greater or equal than $2\sqrt{b^3}$, with equality at $x = 4 + \frac{1}{\sqrt{b}}$. For the initial inequality to have a solution we need $2\sqrt{b^3} \leq \frac{2}{3}b|\cos \pi x|$. Now, as $b \geq 0$ we have $b \leq \frac{1}{9}\cos^2 \pi x$, so that $b \leq \frac{1}{9}$. For $b = \frac{1}{9}$ we have solutions, say $x = 7$.

Answer: $b = \frac{1}{9}$

8. (SPSU) Find all b for which $(a^2 - 9) + 6a \sin x \leq ab$ has solutions for all a.

Answer: $-6 \leq b \leq 6$

9. (MSU) Solve $4 \cos x \sin a + 2 \sin x \cos a - 3 \cos a = 2\sqrt{7}$

Answer: If $a = \arctan \frac{2}{\sqrt{3}} + \frac{\pi}{2} + 2\pi k$ than $x = -\frac{\pi}{6} + 2\pi m$; if $a = -\arctan \frac{2}{\sqrt{3}} - \frac{\pi}{2} + 2\pi k$ than $x = -\frac{5\pi}{6} + 2\pi m$ for $m, k \in \mathrm{Z}$; for other a - no solutions.

Some of the problems look really scary. Yet, all of them are quite solvable.

10. (MSU) Find all a for which the following system has at least one solution:

$$|12\sqrt{\cos\frac{\pi y}{2}} - 5| - |12\sqrt{\cos\frac{\pi y}{2}} - 7| + |24\sqrt{\cos\frac{\pi y}{2}} + 13| = 11 - \sqrt{\sin\frac{\pi(x-2y-1)}{3}},$$

$$2(x^2 + (y-a)^2) - 1 = 2\sqrt{x^2 + (y-a)^2 - \frac{3}{4}}.$$

11. (MSU) For which a does the equation

$$4^{-|x-a|}\log_{\sqrt{3}} x^2 - 2x + 3 + 2^{-(x^2-2x)}\log_{\sqrt{3}} 2|x-a| + 2 = 0$$

has exactly three solutions.

12. (KPI) For which a does $2\cos ax - 3\tan^2 x - 2 = 0$ has exactly one solution?

Answer: a - irrational.

13. Solve $a^5 + x = \sqrt[5]{a - x}$

Graphics: a x coordinates, translation and rotation, families of curves.

Note: In some sense all solutions in this section are not "strict", but they are true, succinct and most of them can be formalized. However, when using these sorts of arguments caution is advisable.

1. (KSU) Find the smallest c for which the system
$$(x - c\sqrt{3})^2 + y^2 - 2y = 0$$
$$\sqrt{3}|x| - y = 4$$
Solutions: Draw a picture. We have a family of circles $(x - c\sqrt{3})^2 + (y - 1)^2 = 1$ and an "angle" $y = \sqrt{3}|x| - 4$. Solving a simple geometry problem now gives an
Answer: $c = -\frac{7}{3}$

2. (MAI) For which a the set of solutions of $\sqrt{5 - x} + \sqrt{x^2 + 2ax + a^2} \leq 3$ is a segment of the real line?
Careful! Answer:$(-8; -\frac{9}{4}] \cup (-2; 4)$.

3. (MSTU) Find all k for which the system
$$\log_2 x + \log_2 y = 6$$
$$y = 8 + k(x - 6)$$
has two different solutions.
Solution: Equivalently
$$xy = 64$$
$$x > 0$$
$$y = 8 + k(x - 6)$$
Now we need the line $y = 8 + k(x - 6)$ to intersect the branch of the hyperbola $xy = 64$ in two points. Finding the slope of the tangents from point $(6, 8)$ to the hyperbola we get the
Answer: $-\frac{9}{4} < k < 0$ or $k < -4$.

4. (MSU) For which a the minimum of $f(x) = ax + |x^2 - 4x + 3|$ is greater than 1.
Answer:$1 < a < 4 + 2\sqrt{2}$

5. (KSU) How many different solutions does the system
$$x^2 + y^2 = 8$$
$$(y - ax)(y - a\sqrt{2}) = 0$$
depending on a?
Answer: If $|a| > 2$ or $a = 0$, then 2 solutions, if $|a| = 2$ or $|a| = \sqrt{3}$ then three, other cases - four solutions.

More Problems

1. (SPSU) For which a does the equation

$$\sqrt[3]{\frac{1}{2}x^2+x+1}+\sqrt[3]{-\frac{1}{2}x^2+x-1}=\sqrt[3]{ax}$$

have exactly four solutions?

2. (MSU) For which p does the system

$$2(p+2y)-y^2=(x-2)^2+z^2$$

$$(xy+4)\sin(x+y)+\cos(x-y)=1$$

$$(2-\frac{xyz(p-2)}{\sqrt{1-2xy}})(p\tan^2 z+x+y)=0$$

has exactly one solution?

3. (MSU) For which a, b does the system

$$|\frac{x^y-1}{x^y+1}|=a$$

$$x^2+y^2=b$$

has exactly one solution?

4. (MSU) For which a does the system

$$(3-2\sqrt{2})^y+(3+2\sqrt{2})^y-3a=x^2+6x+5$$

$$y^2-(a^2-5a+6)x^2=0$$

$$-6\le x\le 0$$

have exactly one solution?

5. (MSU) Find all c such that for any b the system

$$(1+3x^2)^c+(b^2-4b+5)^y=2$$

$$x^2y^2-(2-b)xy+c^2+2c=3$$

has at least one solution.

6. (MSU) Find all x satisfying $\log_{x+a^2+1}(a^2x+2)=2\log_{7+2x}(5-\sqrt{6-2x})$ for any a.

7. (MSU) Find all q for which the equation

$$\sqrt{(|x+2|+q-2\pi+2)(x-3q+20)}+\log_\pi \frac{2\pi^2+q^2}{2(q-\pi)|x+\pi|-x^2-2\pi x+2\pi q}=0$$

has at least on integer solution.

8. (MSU) For which a, b does the equation $|x+\cos^2 4a-2\sin a\cos^4 4a|+|x-\sin^2 a|=b(a+\frac{3\pi}{2})$ has exactly one solution?

9. (MSU) For which a does the system

$$x^2+2xy-7y^2 \geq \frac{1-a}{1+a}$$

$$3x^2+10xy-5y^2 \leq -2$$

has solutions?

Recent MSU entrance exam problems.

1. March 2000

   For which a does

$$\left(\left(\frac{3}{2}\right)^x+\left(\frac{3}{2}\right)^{a-x}-\frac{3}{5}\left(\frac{3}{2}\right)^a-\frac{8}{5}\right)\left(\left(\frac{3}{2}\right)^{2a-2x-3}-4\left(\frac{3}{2}\right)^{2a-5}+2\right)=0$$

   has at least one solution and each solution is an integer?

   Answer:$1, \frac{5}{2}$

2. May 2000

   Find all a for which

$$(2a+4)x^2+(5a+10)x+a+10=0$$

   has 2 solutions and between these solutions there is exactly one solution of

$$(a-1)x^4-(a-1)x^3-(a-7)x^2+(10a+5)x-a+12=0$$

   Answer:$(-\infty,-3)\cup(6,+\infty)$

3. July 2000

   Find all a for which

$$(|a|-1)\cos 2x+(1-|a-2|)\sin 2x+(1-|2-a|)\cos x+(1-|a|)\sin x=0$$

   has odd number of solutions on $(-\pi,\pi)$.

   Answer:$[0,1)\cup(1,2]$

4. March 1999

   For which $a \in [-2;1]$ do the roots of

   $$\sin 2x + |2a+1|\sin x + |a| = 2|a|\cos x + \sin x + |2a^2+a|$$

   lie on the distance not less than $\frac{\pi}{2}$?

   Answer: $-2, 0, 1, \frac{\sqrt{2}}{2}, [-1-\frac{\sqrt{2}}{2}; -1]$.

5. May 1999

   For which a does the set of solutions of

   $$\frac{a+2-2^{x-2}}{a+3} > \frac{5a+5}{2(2^x+3a+3)}$$

   contains a ray on a real line?

   Answer:$a \in (-\infty; -3) \cup \{-1\} \cup [3; +\infty)$

6. July 1999

   For which a is the sum of the lengths of the intervals which are solutions to

   $$\frac{x^2+(2a^2+6)x-(a^2-2a+3)}{x^2+(a^2+7a-7)x-(a^2-2a+3)} < 0$$

   is greater or equal to 1.

   Answer: $a \in (-\infty, 3] \cup [4, +\infty)$

7. March 1998

   For which a does

   $$(x^2+(1-a)x-3(a+2))\log_{x-a}(x-2a-1) = 0$$

   have roots on $[-2, 1]$ and has no solutions outside this interval?

   Answer: $a \in \{-4\} \cup [-3; -1]$

8. May 1998

   For which k at least one common point of $y = -\frac{2}{3} - \arcsin x$ and $y = -\frac{2}{3} - 2\arctan kx$ has positive ordinate?

   Answer: $k \in (\frac{1}{2\cos^2\frac{1}{3}}; 1]$

9. July 1998

   For which α there are exactly 4 solutions to

   $$\cos^2(\pi xy) - 2\sin^2(\pi x) - 3\sin^2(\pi y) - 2 + \tan(\pi\alpha) = 0$$

   $$\cos(\pi xy) - \frac{3}{2}\sin^2(\pi x) - 2\sin^2(\pi y) - \frac{3}{2} + \frac{1}{2}\tan(\pi\alpha) = 0$$

   $$\log_2\left(1 + 4\sin^2\left(\frac{\pi\alpha}{4} - \frac{\pi}{16}\right) - x^2 - y^2\right) \le \frac{1}{2}$$

10. March 1996

    For which a does $2\cos^2(2^{2x-x^2}) = a + \sqrt{3}\sin(2^{2x-x^2+1})$ have at least on solution?

    Answer:$a \in [-1; 2)$

11. May 1996

    For which a is the sum of the roots of

    $$\cos x - \sin 2x + \sin 4x = a(\cot x + 2\cos 3x)$$

    on $[\frac{3\pi}{4}; \frac{22\pi}{3}]$ maximal?

    Answer: $a = -\frac{\sqrt{3}}{2}$

12. May 1996

    For which k there exists b such that $|x^2 - 1| + kx = |x^2 - 8x + 15| + b$ has a) more than 5 b) exactly 5 solutions?

    Answer: a) $k = 8$ b) $k \in (-8; -4\sqrt{3})$.

13. July 1996

    For which a does $(x^3 - x + a^2 + 1)^2 = 4a^2(5x^2 - x + 1)$ have exactly 3 solutions?

    Answer: $a = \pm 1; a = \pm\frac{2+\sqrt{19}}{10}$.

Abbreviations:

KPI - Kiev Polytechnical Institute
KSU - Kiev State University
MAI - Moscow Aviation Institute
MEPHI - Moscow Engineering and Physics Institute
MSTU - Moscow State Technical University
MSU - Moscow State University
SPSU - Saint Petersburg State University