GORDON IDENTITY

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Recall the material about symmetric polynomials and partitions from my previous session.

0.1. **Counting symmetric polynomials.** Set

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n).$$

This is a very convenient notation, which is used in many formulas. For example the Gaussian binomial coefficient and Euler function are given by

$$\binom{m}{r}_q = \frac{(q)_m}{(q)_r (q)_{m-r}}, \qquad \varphi_q = (q)_\infty.$$

Define the counting function of of symmetric polynomials in n-variables by

$$\chi_n(q) = \sum_{i=0}^{\infty} b_{i,n} q^i, \qquad b_{i,n} = \sharp\{\lambda : \ \ \lambda_1 \geqslant \lambda_1 \geqslant \dots \geqslant \lambda_n, \ \ |\lambda| = i\}.$$

The number $b_{i,n}$ counts symmetric polynomials in n variables of total degree i.

Lemma 1.

$$\chi(\mathbb{R}[x_1, \dots, x_n]^S) = \chi_n(q) = \frac{1}{(q)^n},$$

where the geometric progression sum formula, $1/(1-u) = 1 + u + u^2 + \dots$, *is used.*

Exercise: Prove the lemma multiplying out the RHS. □

Exercise: Compute the counting function of the space of all polynomials in n variables $\mathbb{R}[x_1, \dots, x_n]$. □

0.2. **Exact triples.** Suppose we have two spaces of polynomials V_1, V_2 and a map $\alpha: \ V_1 \to V_2$ between them. Assume that α is degree preserving (that is α maps polynomials of total degree n to polynomials of total degree n), linear (maps a sum of two polynomials to the sum of their images) and surjective (any polynomial in V_2 is the image of some polynomial in V_1). Let V_0 is the space of polynomials, which are mapped to 0. Then V_0, V_1, V_3 is called an exact triple and we have

Theorem 2. $\chi(V_1) = \chi(V_0) + \chi(V_2)$.

In other words, the total volume V_1 is the sum of what is left after we transported it by α (e.g. V_2) with what is lost during the transportation (e.g. V_0).

Exercise: Map the space of symmetric polynomials in two variables $V_1 = \mathbb{R}[x, y]^S$ to the space of polynomials in one variable $V_2 = \mathbb{R}[x]$, setting $y = 0$. Show that we have an exact triple with V_0 being the space of symmetric polynomials in x, y divisible by xy. Show that the equality of the theorem has the form

$$\frac{1}{(q)_2} = \frac{q^2}{(q)_2} + \frac{1}{(q)_1}.$$

□

0.3. **Gordon identity.** Let us fix a number n.

For two n-vectors $\nu = (\nu_1, \dots, \nu_n)$, $\mu = (\mu_1, \dots, \mu_n)$, define their Gordon "coupling" $\langle \mu, \nu \rangle$ by the formula

$$\langle \mu, \nu \rangle = \prod_{i,j=1}^{n} \min\{i, j\}\ \nu_i \mu_j .$$

Theorem 3. *(The Gordon Identity)*

$$\frac{q^n}{(q)_n} = \sum_{n = m_1 + 2m_2 + \cdots + km_k} \frac{q^{\langle m, m \rangle}}{(q)_{m_1} (q)_{m_2} \dots (q)_{m_k}} .$$

Exercise: Check the Gordon identity for $n = 1, 2, 3$. □

0.4. **The case $n = 2$.** Map the space of symmetric polynomials in two variables divisible by $x_1 x_2$, $V_1 = x_1 x_2 \mathbb{R}[x_1, x_2]^S$ to the polynomials of one variable divisible by y^2, $V_2 = y^2 \mathbb{R}[y]$, by setting $x_1 = x_2 = y$.

Exercise: Prove that we have an exact triple with V_0 being all symmetric polynomials in two variables divisible by $x_1 x_2 (x_1 - x_2)^2$, $V_0 = x_1 x_2 (x_1 - x_2)^2 \mathbb{R}[x_1, x_2]^S$. Deduce from that the $n = 2$ case of Gordon identity. □

0.5. **The case $n = 3$.** Let F be $x_1 x_2 x_3 \mathbb{R}[x_1, x_2, x_3]^S$. Consider the following subspaces in F.

- F_3 consists of polynomials which are zero if we set $x_1 = x_2 = x_3 = y$.
- $F_{2,1}$ consists of polynomials which are zero if we set $x_1 = x_2 = y$.

We have $F_{2,1} \subset F_3 \subset F$.

Exercise: Prove that $F_{2,1} = x_1 x_2 x_3 \prod_{1 \leqslant i < j \leqslant 3} (x_i - x_j)^2 \mathbb{R}[x_1, x_2, x_3]^S$ and compute the corresponding counting function. □

Exercise: Find an exact triple such that $V_1 = F_3$ and $V_0 = F_{2,1}$ and compute the counting function for F_3. □

Exercise: Find an exact triple such that $V_1 = F$ and $V_0 = F_3$ and deduce from it the $n = 3$ case of the Gordon identity. □

0.6. **The case $n > 3$.** We have spaces F_λ for each partition λ of n which consists of polynomials which are zero if we set the first λ_1 variables equal to y_1, the second λ_2 variables equal to y_2, etc. Then for each λ we have to establish the exact triple with (roughly speaking) $V_0 = F_\lambda$, then the term V_2 will give a contribution to Gordon identity corresponding to the conjugate partition $m = \lambda'$.

Exercise: Identify the first an the last terms in the Gordon identity which correspond to $m = (1, \dots, 1)$ and $m = n$. □

0.7. **So what?** Most of the nontrivial identities in mathematics can be obtained by counting the same set in different ways. Sometimes such counting is not the easiest way to prove an identity, but it is always the most productive way to discover a new complicated identity. It is also usually the best way to memorize an identity.

For example Rogers-Ramanudjan identities can also be described in a similar fashion counting vectors in some other spaces of polynomials.

A simpler example is the numerous identities with binomial coefficients which always can be proved by counting the ways of "choosing".