Symmetric polynomials and partitions

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1. SYMMETRIC POLYNOMIALS

1.1. **Definition.** We will consider polynomials in n variables $x_1, \dots, x_n$ and use the shortcut $p(x)$ instead of $p(x_1, \dots, x_n)$.

A permutation w is a one to one map of the set $\{1, \dots, n\}$ to itself. There are $n!$ permutations. The product of permutations $w_1 w_2$ is just the composition of maps. We will write $w \cdot x$ for $x_{w(1)}, \dots, x_{w(n)}$. An inversion in permutation w is a pair of numbers $1 \leqslant i < j \leqslant n$, such that $w(i) > w(j)$. A permutation w is called even or odd if the number of inversions is even or odd. The sign of a permutation w, $sgn(w)$ is -1 if w is odd and $sgn(w) = 1$ if w is even.

Exercise: Prove that $sgn(w_1 w_2) = sgn(w_2 w_1) = sgn(w_1) sgn(w_2)$. □

Symmetric polynomials are polynomials which do not change values if some arguments are switched.

Definition: A polynomial $p(x)$ is called symmetric if $p(x) = p(w \cdot x)$ for any permutation w.

For example, let $n = 3$, then a polynomial $p(x) = x_1 + x_2 + x_3$ is symmetric, say $p(13, -5, 2) = p(-5, 2, 13)$. The polynomial $q(x) = x_1 + x_2 + x_3 x_1$ is not symmetric, $q(1, 2, 3) \neq q(2, 1, 3)$.

Note that $p(x)$ is the sum of all variables, no matter how you shuffle the variables, but if you permute the variables in q, you can also obtain expressions $x_2 + x_1 + x_3 x_2$, $x_3 + x_2 + x_1 x_2$ and $x_3 + x_1 + x_1 x_2$.

Exercise: Prove that a polynomial $p(x)$ is symmetric if and only if $p(x)$ does not change under the permutations of variables as an expression. □

1.2. **Monomial polynomials.** Let $\lambda = (\lambda_1, \dots, \lambda_n)$.

Definition: The monomial symmetric polynomial m_λ is the sum of monomial $x_1^{\lambda_1} \dots x_n^{\lambda_n}$ and all distinct monomials obtained from it by a permutation of variables.

For example, if $\lambda = (2, 1, 1)$ then $m_\lambda = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$. The total degree of m_λ is $\sum_i \lambda_i$, the degree of m_λ in each variable x_i is λ_1.

In order to avoid repetitions among m_λ we will always assume that $\lambda_1 \geqslant \dots \geqslant \lambda_n$.

A basis is the smallest set of polynomials through which you can express all the others.

Definition: A set of symmetric polynomials S is called a basis, if

1) any symmetric polynomial can be expressed as a sum of polynomials from S with some coefficients.

2) No polynomial from S can be expressed as a sum of other polynomials from S.

Exercise: The monomial polynomials $\{m_\lambda, \lambda = (\lambda_1 \geqslant, \dots \geqslant \lambda_n \geqslant 0)\}$ form a basis. □

1.3. **Partitions. Definition**: The vector $\lambda = (\lambda_1, \dots, \lambda_n)$ is called a partition of k if $\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0$ and $|\lambda| = \lambda_1 + \dots \lambda_n = k$. The number k is called length, numbers λ_i are called parts of λ.

Partitions can be represented by pictures called Young diagrams (or Ferrers diagrams). The Young diagram of λ consists of n rows of boxes aligned on the left, such that i-th row is right on $i+1$-st row. The length of i-th row is λ_i.

The conjugate partition λ' is the partition with the Young diagrams consisting of columns of lengths λ_i. For example λ'_1 is the number of nonzero parts of λ. If $\lambda = (3,3,1)$ then $\lambda' = (3,2,2)$. Also $\lambda'' = \lambda$.

Exercise: Show that the number of partitions of n with odd distinct parts equals to number of self conjugated partitions of n (that is partitions λ with the property $\lambda = \lambda'$). $\square$

Definition: A partition λ is said to be larger than a partition μ if $|\lambda| = |\mu|$ and we have

$$\begin{aligned} \lambda_1 &\geqslant \mu_1 \\ \lambda_1 + \lambda_2 &\geqslant \mu_1 + \mu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\geqslant \mu_1 + \mu_2 + \mu_3 \\ &\cdots \end{aligned}$$

The largest partition of length k is $(k, 0, 0, \dots, 0)$. If $k \leqslant n$ then the smallest partition of length k is $(1, 1, \dots, 1, 0, \dots, 0)$.

Exercise: Show that $\lambda \geqslant \mu$ if and only if the Young diagrams of λ can be obtained from Young diagram of μ by raising some boxes from lower rows to higher ones. $\square$

Exercise: Find an example of two partitions of 6, none of which is greater then another. $\square$

1.4. **Multiplying monomial polynomials.** Let $\mu + \nu$ be a partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_n + \mu_n)$.

Lemma 1.

$$m_\lambda m_\mu = m_{\lambda+\mu} + \sum_{\nu < \lambda+\mu} a^{\nu}_{\lambda,\mu} m_\nu, \qquad a^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geqslant 0}.$$

Exercise: Proof the lemma. $\square$

2. Bases

2.1. **Elementary polynomials.**

Definition: The elementary polynomials e_λ are defined by the formulas

$$\begin{aligned} e_\lambda &= e_{\lambda_1} e_{\lambda_2} \dots, \\ e_r &= m_\lambda \qquad \text{where } \lambda = (1, \dots, 1, 0, \dots, 0) \ (r \text{ ones}). \end{aligned}$$

Lemma 2. *We have*

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu.$$

Therefore $\{e_\lambda,\ \lambda = (n \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0),\ m \in \mathbb{Z}_{\geqslant 0}\}$ *form a basis of symmetric polynomials in* n *variables.*

Exercise: Proof the lemma. □

Note that one can express any symmetric polynomial as a sum of products of e_i, $i = 0, 1, \ldots, n$, where $e_0 = 1$. In the mathematical language $e_1, \ldots, e_n$ are a set of generators of our ring.

2.2. **Power sum polynomials.**

Definition: The power sum polynomials p_λ are defined by the formulas

$$\begin{aligned} p_\lambda &= p_{\lambda_1} p_{\lambda_2} \ldots, \\ p_r &= m_\lambda, \qquad \text{where } \lambda = (r, 0, \ldots, 0) \end{aligned}$$

Lemma 3. *We have*

$$p_\lambda = a_\lambda m_\lambda + \sum_{\mu>\lambda} b_{\lambda\mu} m_m u, \qquad b_{\lambda\mu} \in \mathbb{Z}_{\geqslant 0},$$

where a_λ *is a natural number. Therefore* $\{p_\lambda,\ \lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)\}$ *form a basis of symmetric polynomials.*

Exercise: Proof the lemma. □

2.3. **Complete polynomials.**

Definition: The complete polynomials h_λ are defined by the formulas

$$\begin{aligned} h_\lambda &= h_{\lambda_1} h_{\lambda_2} \ldots, \\ h_r &= \sum_{|\lambda|=r} m_\lambda. \end{aligned}$$

Lemma 4. *Polynomials* $\{h_\lambda,\ \lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)\}$ *form a basis of symmetric polynomials.*

Exercise: Proof the lemma using the relation (1) below. □

2.4. **Schur polynomials.**

Definition: A Schur function s_λ is the sum of function

$$x_1^{\lambda_1} \dots x_n^{\lambda_n} \prod_{i<j} \frac{x_i}{x_i - x_j}$$

with all functions obtained from it by a permutation of variables.

Equivalently, s_λ is the antisymmetrization of monomial $x_1^{\lambda_1} x_2^{\lambda_2+1} \dots x_n^{\lambda_n+n-1}$ divided by the Vandermond function $\prod_{i<j}(x_i - x_j)$,

$$s_\lambda = \left(\sum_w (-1)^{sgn(w)} x_{w(1)}^{\lambda_1} x_2^{\lambda_2+1} \dots x_{w(n)}^{\lambda_n+n-1} \right) / \prod_{i<j}(x_i - x_j),$$

where the sum is over all permutations of n elements.

Exercise: Show that s_λ is a symmetric polynomial. □

Lemma 5.

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu.$$

Therefore $\{s_\lambda,\ \lambda = (\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0)\}$ *form a basis of symmetric polynomials.*

Exercise: Proof the lemma. □

In fact $K_{\lambda\mu}$ are very important nonnegative integers called Kostka numbers.

2.5. **Generating functions and relations between different bases.**

We have the generating functions

$$E(t) := \sum_{i=0}^{n} e_i t^i = \prod_{i=1}^{n} (1 + x_i t),$$

$$H(t) := \sum_{i=0}^{\infty} h_i t^i = \prod_{i=1}^{n} \frac{1}{1 - x_i t},$$

$$P(t) := \sum_{i=1}^{\infty} p_i t^{i-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}.$$

Note that the first equality is a version of Vieta theorem.

We have the relations

$$H(t)E(-t) = 1, \qquad H'(t) = P(t)H(T),$$

therefore

$$\sum_{i=0}^{r} (-1)^s e_i h^{r-i} = 0, \tag{1}$$

$$r h_r = \sum_{i=1}^{r} p_i h_{r-i}.$$

Exercise: Use the relation $P(t) = (\log H(t))'$ to show that

$$h_r = \sum_{|\lambda|=r} \frac{p_\lambda}{z_\lambda}, \qquad z_\lambda = \prod_{i=1}^{n} (i^{m_i} m_i!),$$

where m_i is the number of parts of λ equal i. $\square$

3. Counting symmetric polynomials

3.1. **Gaussian binomial coefficients. Definition**: The Gaussian binomial coefficient is given by

$$\binom{m}{r}_q = \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)}.$$

Exercise: Prove the following identities

$$\binom{m}{r}_1 = \binom{m}{r}, \tag{2}$$

$$\binom{m}{r}_q = q^r \binom{m-1}{r}_q + \binom{m-1}{r-1}_q = \binom{m-1}{r}_q + q^{m-r}\binom{m-1}{r-1}_q, \tag{3}$$

$$\prod_{i=0}^{n-1}(1+q^i t) = \sum_{i=0}^{n} q^{i(i-1)/2}\binom{n}{i}_q t^i, \tag{4}$$

$$\prod_{i=0}^{n-1}\frac{1}{1-q^i t} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q t^i. \tag{5}$$

The identity 2 shows that Gaussian binomial coefficients are generalizations of usual binomial coefficients. The identities 3 are called Pascal idenitites, the identity 4 is called Newton binomial formula. Use one of the identities to show that Gaussian binomial coefficient is a polynomial in q. $\square$

3.2. **Main theorem via recursion relations.** Define the counting function of of symmetric polynomials by

$$\chi_{n,k}(q) = \sum_{i=0}^{\infty} a_{i,k} q^i, \qquad a_{i,k} = \sharp\{\lambda,\ \ \lambda_1 \leqslant k,\ \ |\lambda| = i\}.$$

The number $a_{i,k}$ counts polynomials of total degree i, such that degree in any variable is at most k.

Theorem 6.

$$\chi_{k,n}(q) = \binom{n+k}{k}_q.$$

Exercise: Prove the theorem using Pascal identity (3). $\square$

3.3. **Main theorem via h_k.** Note that LHS of (5) is equal to $H(t)$ where x_i are substituted with q^{i-1}.

Exercise: Prove Theorem 6 by comparing monomials in h_k with $n+1$ variables and partitions λ contributing to a_i. □

4. Appendix

4.1. **Euler Identity.** The Euler function is defined by the formula

$$\varphi(t)=\prod_{i=1}^{\infty}(1-t^i).$$

The coefficient of t^k in function $1/\varphi(t)$ equals the number of all partitions of k.

Exercise: Use generating functions to prove that the number of partitions of n with odd parts is equal to the number of partitions of n with unequal parts. □

Lemma 7. *(Euler identity)*

$$\varphi(t)=\sum_{n=-\infty}^{\infty}(-1)^i t^{(3n^2-n)/2}.$$

The numbers $(3n^2-n)/2$ are called pentagon numbers. Compare to numbers n, triangular numbers $n(n+1)/2$, square numbers n^2.

Exercise: Prove Euler identity by constructing a map from partitions consisting of odd number of unequal parts to partitions consisting of even number of unequal parts. □

4.2. **Rogers–Ramanudjan Identities.**

Lemma 8.

$$\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^7}\frac{1}{1-q^8}\cdots=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$\frac{1}{1-q^1}\frac{1}{1-q^4}\frac{1}{1-q^6}\frac{1}{1-q^9}\cdots=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Here in LHS of the first identity we have powers of q which have remainders 2 or 3 mod 5 and in LHS of the second identity the powers have remainders 1 or 4 mod 5.

Exercise: Reformulate Rogers-Ramanudjan identities in the language of partitions. □

4.3. **A challenge.** Let N_k be a number of different figures obtained by a putting two Young diagrams of partitions λ, μ, such that $|\lambda|+|\mu|=k$ on top of each other. For example, $N_0=N_1=1$, $N_2=3$, $N_4=5$, $N_5=10$, $N_6=16$.

CHALLENGE. Compute the function $N(t)=\sum_{i=0}^{\infty}N_i t^i$.

At the moment I know the answer but I do not know an elementary proof of it.