1. Symmetric polynomials

1.1. Definition. We will consider polynomials in \( n \) variables \( x_1, \ldots, x_n \) and use the shortcut \( p(x) \) instead of \( p(x_1, \ldots, x_n) \).

A permutation \( w \) is a one to one map of the set \( \{1, \ldots, n\} \) to itself. There are \( n! \) permutations. The product of permutations \( w_1w_2 \) is just the composition of maps. We will write \( w \cdot x \) for \( x_{w(1)}, \ldots, x_{w(n)} \). An inversion in permutation \( w \) is a pair of numbers \( 1 \leq i < j \leq n \), such that \( w(i) > w(j) \). A permutation \( w \) is called even or odd if the number of inversions is even or odd. The sign of a permutation \( w \), \( \text{sgn}(w) \) is \(-1\) if \( w \) is odd and \( \text{sgn}(w) = 1 \) if \( w \) is even.

Exercise: Prove that \( \text{sgn}(w_1w_2) = \text{sgn}(w_2w_1) = \text{sgn}(w_1)\text{sgn}(w_2) \). \( \square \)

Symmetric polynomials are polynomials which do not change values if some arguments are switched.

Definition: A polynomial \( p(x) \) is called symmetric if \( p(x) = p(w \cdot x) \) for any permutation \( w \).

For example, let \( n = 3 \), then a polynomial \( p(x) = x_1 + x_2 + x_3 \) is symmetric, say \( p(13, -5, 2) = p(-5, 2, 13) \). The polynomial \( q(x) = x_1 + x_2 + x_3x_1 \) is not symmetric, \( q(1, 2, 3) \neq q(2, 1, 3) \).

Note that \( p(x) \) is the sum of all variables, no matter how you shuffle the variables, but if you permute the variables in \( q \), you can also obtain expressions \( x_2 + x_1 + x_3x_2, x_3 + x_2 + x_1x_2 \) and \( x_3 + x_1 + x_1x_2 \).

Exercise: Prove that a polynomial \( p(x) \) is symmetric if and only if \( p(x) \) does not change under the permutations of variables as an expression. \( \square \)

1.2. Monomial polynomials. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

Definition: The monomial symmetric polynomial \( m_\lambda \) is the sum of monomial \( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \) and all distinct monomials obtained from it by a permutation of variables.

For example, if \( \lambda = (2, 1, 1) \) then \( m_\lambda = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 \). The total degree of \( m_\lambda \) is \( \sum_i \lambda_i \), the degree of \( m_\lambda \) in each variable \( x_i \) is \( \lambda_1 \).

In order to avoid repetitions among \( m_\lambda \) we will always assume that \( \lambda_1 \geq \cdots \geq \lambda_n \).

A basis is the smallest set of polynomials through which you can express all the others.

Definition: A set of symmetric polynomials \( S \) is called a basis, if

1) any symmetric polynomial can be expressed as a sum of polynomials from \( S \) with some coefficients.

2) No polynomial from \( S \) can be expressed as a sum of other polynomials from \( S \).

Exercise: The monomial polynomials \( \{m_\lambda, \lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)\} \) form a basis. \( \square \)
1.3. **Partitions. Definition:** The vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called a partition of $k$ if $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $|\lambda| = \lambda_1 + \cdots + \lambda_n = k$. The number $k$ is called length, numbers $\lambda_i$ are called parts of $\lambda$.

Partitions can be represented by pictures called Young diagrams (or Ferrers diagrams). The Young diagram of $\lambda$ consists of $n$ rows of boxes aligned on the left, such that $i$-th row is right on $i+1$-st row. The length of $i$-th row is $\lambda_i$.

The conjugate partition $\lambda'$ is the partition with the Young diagrams consisting of columns of lengths $\lambda_i$. For example $\lambda'_1$ is the number of nonzero parts of $\lambda$. If $\lambda = (3, 3, 1)$ then $\lambda' = (3, 2, 2)$. Also $\lambda'' = \lambda$.

**Exercise:** Show that the number of partitions of $n$ with odd distinct parts equals to number of self conjugated partitions of $n$ (that is partitions $\lambda$ with the property $\lambda = \lambda'$).

**Definition:** A partition $\lambda$ is said to be larger than a partition $\mu$ if $|\lambda| = |\mu|$ and we have

\[
\begin{align*}
\lambda_1 & \geq \mu_1 \\
\lambda_1 + \lambda_2 & \geq \mu_1 + \mu_2 \\
\lambda_1 + \lambda_2 + \lambda_3 & \geq \mu_1 + \mu_2 + \mu_3 \\
& \vdots
\end{align*}
\]

The largest partition of length $k$ is $(k, 0, 0, \ldots, 0)$. If $k \leq n$ then the smallest partition of length $k$ is $(1, 1, \ldots, 1, 0, \ldots, 0)$.

**Exercise:** Show that $\lambda \geq \mu$ if and only if the Young diagrams of $\lambda$ can be obtained from Young diagram of $\mu$ by raising some boxes from lower rows to higher ones.

**Exercise:** Find an example of two partitions of 6, none of which is greater than another.

1.4. **Multiplying monomial polynomials.** Let $\mu + \nu$ be a partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_n + \mu_n)$.

**Lemma 1.**

\[
m_\lambda m_\mu = m_{\lambda+\mu} + \sum_{\nu<\lambda+\mu} a^\nu_{\lambda,\mu} m_\nu,
\]

where $a^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$.

**Exercise:** Proof the lemma.

---

2. **Bases**

2.1. **Elementary polynomials.**

**Definition:** The elementary polynomials $e_\lambda$ are defined by the formulas

\[
en_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots,
\]

\[
ere = m_\lambda \quad \text{where} \quad \lambda = (1, \ldots, 1, 0, \ldots, 0) \ (r \ \text{ones}).
\]
Lemma 2. We have
\[ e_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda \mu} m_\mu. \]

Therefore \( \{e_\lambda, \lambda = (n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0), m \in \mathbb{Z}_{\geq 0}\} \) form a basis of symmetric polynomials in \( n \) variables.

Exercise: Proof the lemma. □

Note that one can express any symmetric polynomial as a sum of products of \( e_i, i = 0, 1, \ldots, n \), where \( e_0 = 1 \). In the mathematical language \( e_1, \ldots, e_n \) are a set of generators of our ring.

2.2. Power sum polynomials.

Definition: The power sum polynomials \( p_\lambda \) are defined by the formulas
\[
\begin{align*}
p_\lambda &= p_{\lambda_1}p_{\lambda_2} \cdots, \\
p_r &= m_\lambda, \quad \text{where } \lambda = (r, 0, \ldots, 0)
\end{align*}
\]

Lemma 3. We have
\[
p_\lambda = a_\lambda m_\lambda + \sum_{\mu > \lambda} b_{\lambda \mu} m_\mu, \quad b_{\lambda \mu} \in \mathbb{Z}_{\geq 0},
\]

where \( a_\lambda \) is a natural number. Therefore \( \{p_\lambda, \lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)\} \) form a basis of symmetric polynomials.

Exercise: Proof the lemma. □

2.3. Complete polynomials.

Definition: The complete polynomials \( h_\lambda \) are defined by the formulas
\[
\begin{align*}
h_\lambda &= h_{\lambda_1}h_{\lambda_2} \cdots, \\
h_r &= \sum_{|\lambda|=r} m_\lambda.
\end{align*}
\]

Lemma 4. Polynomials \( \{h_\lambda, \lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)\} \) form a basis of symmetric polynomials.

Exercise: Proof the lemma using the relation (1) below. □
2.4. Schur polynomials.

**Definition:** A Schur function $s_\lambda$ is the sum of function

$$x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{i<j} \frac{x_i}{x_i - x_j}$$

with all functions obtained from it by a permutation of variables.

Equivalently, $s_\lambda$ is the antisymmetrization of monomial $x_1^{\lambda_1} x_2^{\lambda_2+1} \ldots x_n^{\lambda_n+n-1}$ divided by the Vandermond function $\prod_{i<j} (x_i - x_j)$,

$$s_\lambda = \left( \sum_w (-1)^{sgn(w)} x_1^{\lambda_1} x_2^{\lambda_2+1} \ldots x_n^{\lambda_n+n-1} \right) / \prod_{i<j} (x_i - x_j),$$

where the sum is over all permutations of $n$ elements.

**Exercise:** Show that $s_\lambda$ is a symmetric polynomial. □

**Lemma 5.**

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu.$$ 

Therefore $\{s_\lambda, \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)\}$ form a basis of symmetric polynomials.

**Exercise:** Proof the lemma. □

In fact $K_{\lambda\mu}$ are very important nonnegative integers called Kostka numbers.

2.5. Generating functions and relations between different bases. We have the generating functions

$$E(t) := \sum_{i=0}^{n} e_i t^i = \prod_{i=1}^{n} (1 + x_i t),$$

$$H(t) := \sum_{i=0}^{\infty} h_i t^i = \prod_{i=1}^{n} \frac{1}{1 - x_i t},$$

$$P(t) := \sum_{i=1}^{\infty} p_i t^{i-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}.$$ 

Note that the first equality is a version of Vieta theorem.

We have the relations

$$H(t) E(-t) = 1, \quad H'(t) = P(t) H(T),$$

therefore

$$\sum_{i=0}^{r} (-1)^s e_i h^{r-i} = 0, \quad (1)$$

$$r h_r = \sum_{i=1}^{r} p_i h_{r-i}.$$
**Exercise:** Use the relation \( P(t) = (\log H(t))' \) to show that
\[
h_r = \sum_{|\lambda|=r} \frac{p_\lambda}{z_\lambda}, \quad z_\lambda = \prod_{i=1}^n (i^{m_i} m_i!)
\]
where \( m_i \) is the number of parts of \( \lambda \) equal \( i \).

3. **Counting symmetric polynomials**

3.1. **Gaussian binomial coefficients. Definition:** The Gaussian binomial coefficient is given by
\[
\begin{align*}
\binom{m}{r}_q &= \frac{(1 - q^m)(1 - q^{m-1}) \ldots (1 - q^{m-r+1})}{(1 - q)(1 - q^2) \ldots (1 - q^r)}.
\end{align*}
\]

**Exercise:** Prove the following identities
\[
\begin{align*}
\binom{m}{r}_1 &= \binom{m}{r}, \\
\binom{m}{r}_q &= q^{r} \binom{m-1}{r}_q + \binom{m-1}{r-1}_q = \binom{m-1}{r}_q + q^{m-r} \binom{m-1}{r-1}_q, \\
\prod_{i=0}^{n-1} (1 + q^it) &= \sum_{i=0}^{n} q^{(i-1)/2} \binom{n}{i}_q t^i, \\
\prod_{i=0}^{n-1} \frac{1}{1 - q^i t} &= \sum_{i=0}^{\infty} \binom{n + i - 1}{i}_q t^i.
\end{align*}
\]

The identity 2 shows that Gaussian binomial coefficients are generalizations of usual binomial coefficients. The identities 3 are called Pascal identities, the identity 4 is called Newton binomial formula. Use one of the identities to show that Gaussian binomial coefficient is a polynomial in \( q \).

3.2. **Main theorem via recursion relations.** Define the counting function of of symmetric polynomials by
\[
\chi_{n,k}(q) = \sum_{i=0}^{\infty} a_{i,k} q^i, \quad a_{i,k} = \sharp \\{ \lambda, \lambda_1 \leq k, |\lambda| = i \}.
\]

The number \( a_{i,k} \) counts polynomials of total degree \( i \), such that degree in any variable is at most \( k \).

**Theorem 6.**
\[
\chi_{k,n}(q) = \binom{n+k}{k}_q.
\]

**Exercise:** Prove the theorem using Pascal identity (3).
3.3. **Main theorem via** $h_k$. Note that LHS of (5) is equal to $H(t)$ where $x_i$ are substituted with $q^{-i}$.

**Exercise**: Prove Theorem 6 by comparing monomials in $h_k$ with $n + 1$ variables and partitions $\lambda$ contributing to $a_i$. □

4. **Appendix**

4.1. **Euler Identity**. The Euler function is defined by the formula

$$\varphi(t) = \prod_{i=1}^{\infty} (1 - t^i).$$

The coefficient of $t^k$ in function $1/\varphi(t)$ equals the number of all partitions of $k$.

**Exercise**: Use generating functions to prove that the number of partitions of $n$ with odd parts is equal to the number of partitions of $n$ with unequal parts. □

**Lemma 7.** (*Euler identity*)

$$\varphi(t) = \sum_{n=-\infty}^{\infty} (-1)^{i}t^{(3n^2-n)/2}.$$

The numbers $(3n^2 - n)/2$ are called pentagon numbers. Compare to numbers $n$, triangular numbers $n(n+1)/2$, square numbers $n^2$.

**Exercise**: Prove Euler identity by constructing a map from partitions consisting of odd number of unequal parts to partitions consisting of even number of unequal parts. □

4.2. **Rogers–Ramanujan Identities.**

**Lemma 8.**

$$\frac{1}{1 - q^2} \frac{1}{1 - q^3} \frac{1}{1 - q^7} \frac{1}{1 - q^8} \cdots = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1 - q)(1 - q^2) \ldots (1 - q^n)},$$

$$\frac{1}{1 - q} \frac{1}{1 - q^4} \frac{1}{1 - q^6} \frac{1}{1 - q^9} \cdots = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \ldots (1 - q^n)}.$$

Here in LHS of the first identity we have powers of $q$ which have remainders 2 or 3 mod 5 and in LHS of the second identity the powers have remainders 1 or 4 mod 5.

**Exercise**: Reformulate Rogers-Ramanujan identities in the language of partitions. □

4.3. **A challenge.** Let $N_k$ be a number of different figures obtained by a putting two Young diagrams of partitions $\lambda, \mu$, such that $|\lambda| + |\mu| = k$ on top of each other. For example, $N_0 = N_1 = 1$, $N_2 = 3$, $N_4 = 5$, $N_5 = 10$, $N_6 = 16$.

**CHALLENGE.** Compute the function $N(t) = \sum_{i=0}^{\infty} N_i t^i$.

At the moment I know the answer but I do not know an elementary proof of it.